Open Problems in Discrete Differential Geometry
Collected by Günter Rote

PROBLEM 1 (Sergei Tabachnikov). Paper Möbius strip and paper cylinder eversion
One can make a smooth Möbius strip from a paper rectangle if its aspect ratio is sufficiently large, but not from a square.

Question 1. *What is the smallest length λ such that a smooth Möbius band can be made of a $1 \times a$ paper rectangle if $a > \lambda$?*

The known bounds are $\frac{\pi}{2} \leq \lambda \leq \sqrt{3}$ [1, 2], and it is conjectured that $\lambda = \sqrt{3}$. For smooth immersions, the answer is $\lambda = \pi/2$. See [3, 4, 5] and the references there for developable Möbius bands. A related problem concerns the eversion of a cylinder:

Question 2. *What is the least perimeter μ of a paper cylinder of height* 1 *that can be turned inside out in the class of embedded smooth developable surfaces.*

The known bounds are $\pi \leq \mu \leq \pi + 2$, and for smooth immersions, the answer is $\mu = \pi$ [2].

[1] D. Fuchs, S. Tabachnikov. *Mathematical omnibus. Thirty lectures on classic mathematics.* Amer. Math. Soc., Providence, RI, 2007.
[2] B. Halpern, C. Weaver. *Inverting a cylinder through isometric immersions and isometric embeddings.* Trans. Amer. Math. Soc. 230 (1977), 41–70.
[3] T. Randrup, P. Røgen. *Sides of the Möbius strip.* Arch. Math. 66 (1996), 511–521.
[4] I. Sabitov. *Isometric immersions and embeddings of a flat Möbius strip into Euclidean spaces.* Izv. Math. 71 (2007), 1049–1078.
[5] G. Schwarz. *The dark side of the Moebius strip.* Amer. Math. Monthly 97 (1990), 890–897.

PROBLEM 2 (Sergei Tabachnikov). Commuting billiard maps
Given a smooth convex plane domain, the billiard ball map sends an incoming ray (the trajectory of the billiard ball) that hits the boundary from inside to an outgoing ray according to the law of reflection: the angle of incidence equals the angle of reflection.

Consider two nested convex domains. The two billiard ball maps, T_1 and T_2, act on the oriented lines that intersect both domains. If the domains are bounded by confocal ellipses, then the respective billiard ball maps commute; see, e.g., [4].

Question. *Assume that the two maps commute: $T_1 \circ T_2 = T_2 \circ T_1$. Does it follow that the two domains are bounded by confocal ellipses?*

For piecewise analytic billiards, this conjecture was proved in [2]. For "outer billiards", an analogous fact is proved in [3]. Of course, this problem has a multi-dimensional version, open both for inner and outer billiards; see, e.g., [1] on multi-dimensional integrable billiards.

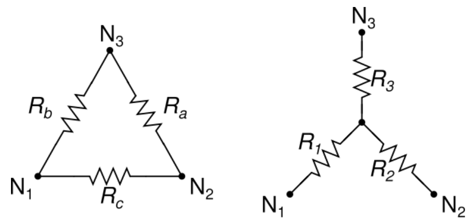
[1] V. Dragović, M. Radnović. *Poncelet porisms and beyond. Integrable billiards, hyperelliptic Jacobians and pencils of quadrics.* Birkhäuser, Basel, 2011.
[2] A. Glutsyuk. *On 4-reflective complex analytic planar billiards.* arXiv:1405.5990, 2014.
[3] S. Tabachnikov. *Commuting dual billiard maps.* Geom. Dedicata 53 (1994), 57–68.
[4] S. Tabachnikov. *Geometry and billiards.* Amer. Math. Soc., Providence, RI, 2005.

PROBLEM 3 (Mikhail Skopenkov). INVERSE PROBLEM FOR ALTERNATING-CURRENT NETWORKS

An *alternating-current* network [2, Section 2.4] is a (not necessarily planar) graph with a fixed subset of b vertices (*boundary vertices*) and a complex number c_{xy} with positive real part (*conductance*) assigned to each edge xy. A *voltage* is a complex-valued function v_x on the set of vertices such that for each nonboundary vertex y we have $\sum_{xy} c_{xy}(v_x - v_y) = 0$, where the sum is over the edges containing the vertex y. One can see that the voltage is uniquely determined by its boundary values [2, Section 5.1]. The *current flowing into the network* through a boundary vertex y is $i(y) := \sum_{xy} c_{xy}(v_x - v_y)$. The network *response* is the matrix of the linear map taking the vector of voltages at the boundary vertices to the vector of currents flowing into the network through the boundary vertices.

The general *electrical-impedance tomography* problem is to reconstruct the network from its response. For direct-current planar networks, meaning that all conductances are real and positive, the problem has been solved [1].

Teaser. There is a matrix realizable as the response of the network in the figure to the right, for the boundary vertices N_1, N_2, N_3 and some edge conductances R_1, R_2, R_3, but not by the network to the left.



Denote by Ψ_b the set of complex $b \times b$ matrices Λ having the following 4 properties:

(1) Λ is symmetric;
(2) the sum of the entries of Λ in each row is zero;
(3) $\operatorname{Re}\Lambda$ is positive semidefinite;
(4) if $U = (U_1, \dots, U_b) \in \mathbb{R}^b$ and $U^T(\operatorname{Re}\Lambda)U = 0$ then $U_1 = \dots = U_b$.

Question 1. *Prove that the set of responses of all possible connected alternating-current networks with b boundary vertices is the set Ψ_b.*

It is known that Conditions (1–4) are necessary. Sufficiency is known for $b = 2$ and $b = 3$ [2, Theorem 4.7].

Question 2. *Provide an algorithm to reconstruct a network and edge conductances for a given response matrix.*

Note: Questions 1 and 2 have been solved by Günter Rote.

Question 3. *Describe the set of responses of all* series-parallel *networks.*

Question 4. *Describe the set of responses of all* planar *networks that have the boundary vertices on the outer face.*

Question 5. *Let the conductance of each edge be either ω or $1/\omega$, where ω is a variable. Describe the set of possible responses of such networks as functions in ω.*

This is known for $b = 2$ boundary vertices — Foster's reactance theorem [2, Theorem 2.5].

[1] E. B. Curtis, J. A. Morrow, *Inverse Problems for Electrical Networks*, Ser. Appl. Math., vol. 13, World Scientific, Singapore, 2000.

[2] M. Prasolov and M. Skopenkov, *Tiling by rectangles and alternating current*, J. Combin. Theory A 118:3 (2011), 920–937.

PROBLEM 4 (Nina Amenta). Are face angles determined by dihedral angles?

Stoker's conjecture says that, for a convex 3-polytope with given combinatorics, if all dihedral angles are specified (different from 0 and π), then all face angles are also determined.

Question. *Is this also true for a non-convex polytope?*

One may assume that the polytope is triangulated and that it is homeomorphic to a sphere.

PROBLEM 5 (Günter Rote). Existence of offset polytopes

We are given a non-convex three-dimensional polytope P whose boundary is homeomorphic to a sphere. We want to construct an *offset* polytope P_ε in which every face is translated outward by the same small distance ε. P_ε should not have other faces than the faces coming from P, and the boundary of P_ε should remain homeomorphic to a sphere. If P has a saddle-like vertex of degree 4 or larger, the result is not unique.

Question. *Does such an offset polytope always exist for sufficiently small $\varepsilon > 0$?*

It is enough to solve the problem locally for each vertex v of degree $d \geq 4$. Such a vertex will be blown up into $d - 2$ new vertices, connected by edges that form a tree. The faces of P_ε should be simply connected when clipped to a neighborhood of v.

PROBLEM 6 (Ulrich Bauer). Subdivision of discrete conformal structures

Let $\mathcal{T}$ be a triangulated surface, and let $\lambda, \mu \colon E_{\mathcal{T}} \to \mathbb{R}_{>0}$ be two discrete conformally equivalent metrics on $\mathcal{T}$, represented as edge lengths of the triangulation. Here, discrete conformal equivalence means that the edge lengths of λ and μ for any edge ij between two vertices i, j are related by $\mu_{ij} = e^{\frac{1}{2}(u_i+u_j)}\lambda_{ij}$ for some function $u \colon V_{\mathcal{T}} \to \mathbb{R}$ on the vertices.

Question. *Is there a metric subdivision scheme that preserves the conformal equivalence?*

Specifically, a *subdivision* of a simplicial complex K is a complex K' such that $|K| = |K'|$ and each simplex of K' is contained in some simplex of K. A *metric subdivision scheme* is a map sending each simplicial complex K equipped with a metric λ to a subdivision K' of K equipped with a metric λ'. A particular example is the *barycentric subdivision.* The question is whether there exists a metric subdivision scheme such that the subdivided metrics λ' and μ' are still conformally equivalent.

[1] F. Luo. *Combinatorial Yamabe flow on surfaces.* Commun. Contemp. Math., 6(5):765–780, 2004.

PROBLEM 7 (Jim Propp, Richard Kenyon). DISK PACKINGS OF MAXIMUM AREA

Consider two disks of radius 1 with centers at $(\pm 1, 1)$. Together with the x-axis, they enclose at curved triangular region. Into this region, we want to place infinitely many non-overlapping disks that touch the x-axis, such that their total area is maximized.

Question. *Is it true that the greedy method of successively placing each new circle into the interstices such that they touch two previously placed circles will give the maximum area?*

PROBLEM 8 (Günter Rote). A CURIOUS IDENTITY ON SELF-STRESSES

Take the wheel graph G (the graph of a pyramid) embedded in the plane in general position, with a central vertex p_0 that is connected to vertices $p_1, \dots, p_n$ $(n \geq 3)$ forming a cycle. On the $2n$ edges of this graph, we define the following function:

$$\omega_{i,i+1} := \frac{1}{[p_i p_{i+1} p_0][p_1 p_2 \dots p_n]}, \qquad \omega_{0,i} := \frac{[p_{i-1} p_i p_{i+1}]}{[p_{i-1} p_i p_0][p_i p_{i+1} p_0][p_1 p_2 \dots p_n]}.$$

for $i = 1, \dots, n$, where $[q_1 q_2 \dots q_k]$ denotes signed area of the polygon $q_1 q_2 \dots q_k$, and $p_{n+1} = p_1$. This function is a *self-stress*: the equilibrium condition $\sum_j \omega_{ij}(p_j - p_i) = 0$ holds for every vertex i, where the summation is over all edges ij incident to i. Pick two arbitrary points a and b and define another function f_{ij} on the edges of G:

$$f_{ij} := [a p_i p_j][b p_j p_j],$$

Then we have the following identity, which was used (and proved) for $n = 3$ in [1].

$$\sum_{ij \in E(G)} \omega_{ij} f_{ij} = 1 \tag{1}$$

A different formula for f_{ij} that fulfills (1) is given by a line integral over the segment $p_i p_j$, see [1, Lemma 3.10]:

$$f'_{ij} := \tfrac{3}{2} \cdot \|p_i - p_j\| \cdot \int_{x=p_i}^{p_j} \|x\|^2 \, ds = \tfrac{1}{2} \cdot \|p_i - p_j\|^2 \cdot \left(\|p_i\|^2 + \|p_j\|^2 + \langle p_i, p_j \rangle \right)$$

Question 1. *Are there other graphs with n vertices and $2n-2$ edges, for which a self-stress ω satisfying* (1) *can be defined? The next candidates with* 6 *vertices are the graph of a triangular prism with an additional edge, and the complete bipartite graph $K_{3,3}$ with an additional edge.*

Question 2. *What is the meaning of the identity* (1)*? Is it an instance of a more general phenomenon? What are the connections to homology and cohomology?*

[1] Günter Rote, Francisco Santos, and Ileana Streinu, *Expansive motions and the polytope of pointed pseudo-triangulations*, in: Discrete and Computational Geometry—The Goodman-Pollack Festschrift, Springer, 2003, pp. 699–736, arXiv:math/0206027 [math.CO].

PROBLEM 9 (Hao Chen and Arnau Padrol, reported by Günter M. Ziegler).
APPROXIMATELY INSCRIBED POLYTOPES

Steinitz proved in 1928 [3] that not every combinatorial type of 3-polytope can be inscribed, that is, realized with all vertices on a sphere. However, a weak version of this is true: Due to the Koebe–Andreev–Thurston circle packing theorem (see e.g. [2, 4]), every 3-polytope can be realized with all edges tangent to the sphere – and thus it has a representation with

- all vertices outside a sphere
- all facets cutting into the sphere.

The question is whether this extends to higher dimensions:

Question. *Does every combinatorial type of d-polytope have a realization with*

- *vertices outside a $(d-1)$-sphere,*
- *facets cutting into the same $(d-1)$-sphere.*

Our conjecture is that this is false for $d > 3$, perhaps already for $d = 4$, but certainly for high dimensions d, where we know that there are infinitely many projectively unique polytopes. The examples constructed in [1] are essentially inscribable, but there should be other such polytopes whose "shape" is far off from that of a sphere/quadric. However, we have not been able to construct such a polytope yet.

[1] Karim Adiprasito and Günter M. Ziegler. *Many projectively unique polytopes.* Inventiones math., 199:581–652, 2015.

[2] Alexander I. Bobenko and Boris A. Springborn. *Variational principles for circle patterns, and Koebe's theorem.* Transactions Amer. Math. Soc., 356:659–689, 2004.

[3] Ernst Steinitz. *Über isoperimetrische Probleme bei konvexen Polyedern.* J. reine angewandte Math., 159:133–143, 1928.

[4] Günter M. Ziegler. *Convex polytopes: Extremal constructions and f-vector shapes.* In E. Miller, V. Reiner, and B. Sturmfels, editors, "Geometric Combinatorics", Proc. Park City Mathematical Institute (PCMI) 2004, pages 617–691, Providence, RI, 2007. Amer. Math. Society.