Discrete Geometry Workshop - Oberwolfach 2017
Open Problems
Collected by J. Pach and S. Zerbib

1. Andrey Kupavskii, kupavskii@ya.ru

"Given n slabs in $\mathbb{R}^d$ of total divergent width, can one cover the unit ball with their translates?"
In more details: is it true that there exists $C = C(d)$, such that for any $n_1, \ldots, n_s \in S^{d-1}$, $d > 2$, and any $\varepsilon_1, \ldots, \varepsilon_s \in \mathbb{R}_+$ with $\sum_{i=1}^s \varepsilon_i > C$, there exist $x_1, \ldots, x_s \in \mathbb{R}$ satisfying

$$\{x \in \mathbb{R}^d : \ |x| \leq 1\} \subseteq \{x \in \mathbb{R}^d : \ x_i \leq \langle x, n_i \rangle \leq x_i + \varepsilon_i\}?$$

Asked in [Makai–Pach, 1983].

E. Makai and J. Pach, Controlling function classes and covering Euclidean space, *Studia Sci. Math. Hungar.* **18** (1983), no. 2–4, 435–459.

2. Dömötör Pálvölgyi, dom@cs.elte.hu

Can we 3-color any (finite) set of points such that any disk with at least 3 points is non-monochromatic? Asked originally in [Keszegh, 2012].

B. Keszegh, Coloring half-planes and bottomless rectangles, *Computational Geometry: Theory and Applications*, **45(9)** (2012), 495–507.

3. Eran Nevo, nevo@math.huji.ac.il

Fix d even, and let $n \to \infty$:
Must d-polytopes with n vertices have only $o\big(n^{d/2}\big)$ non-simplex facets? (The trivial upper bound is $O\big(n^{d/2}\big)$.)

Jeff Erickson asked this in 1999, and conjectured that the answer is yes, also for $(d-1)$-polyhedral spheres.
For spheres the answer is no - as was proved in [Nevo–Santos–Wilson, 2016]

The case $d = 4$ of the above question is already very interesting. The lower bound obtained in Nevo *et al.* is $\Omega(n^{3/2})$.

E. Nevo, F. Santos and S. Wilson, Many triangulated odd-dimensional spheres, *Math. Ann.* **364** (2016), no. 3–4, 737–762.

4. Arseniy Akopyan, akopjan@gmail.com

Let P_1 and P_2 be two combinatorially equivalent convex polytopes in $\mathbb{R}^3$. Is it true that there exist corresponding edges t_1 of P_1 and t_2 of P_2, such that the dihedral angle of t_1 is not greater than the dihedral angle of t_2, or all the corresponded angles are equal? This problem is Conjecture 5.1 in the following preprint.

A. V. Akopyan and R. N. Karasev, *Bounding minimal solid angles of polytopes*, 2015, http://arxiv.org/abs/1505.05263.

5. Micha Sharir, michas@post.tau.ac.il

Danzer's problem. A finite set of pairwise intersecting disks in the plane can be stabbed by 4 points, and there exists a configuration of 10 pairwise intersecting disks that require 4 points [Danzer, 1986].

The problem:
(a) Understand Danzer's solution.
(b) Come up with a simpler solution.
(c) Make it constructive.

L. Danzer, Zur Lösung des Gallaischen Problems über Kreisscheiben in der Euklidischen Ebene [On the solution of the Gallai problem on circular disks in the Euclidean plane, in German], *Studia Sci. Math. Hungar.* **21** (1986), no. 1–2, 111–134.

6. Xavier Goaoc, goaoc@univ-mlv.fr

Fact: For any probability measure μ that charges no lines, there exist two order types $\omega_1(\mu)$ and $\omega_2(\mu)$ of size 6 such that if X is a set of 6 points $\sim \mu$ then

$$\mathbb{P}[X \text{ realizes } \omega_1(\mu)] > 1.8\ \mathbb{P}[X \text{ realizes } \omega_2(\mu)].$$

X. Goaoc, A. Hubard, R. de Joannis de Verclos, J-S. Sereni, and J. Volec, Limits of order types, *Proceedings of Symp. of Computational Geometry* (SOCG), vol 34, pp 300-314, 2015.

Question: Does there exist $c > 0$ such that $\forall \mu \, \exists \, \omega_1(\mu), \; \omega_2(\mu)$ with $|\omega_1(\mu)| = |\omega_2(\mu)| = n$ and

$$\mathbb{P}[X \simeq \omega_1] > c^n \; \mathbb{P}[X \simeq \omega_2(\mu)]?$$

7. Géza Tóth, geza@renyi.hu

Is the class of intersection graphs of lines in $\mathbb{R}^3$ (or $\mathbb{R}^d$) χ-bounded? Namely, is there a function f such that given n lines in the $\mathbb{R}^3$, no k of them pairwise crossing, the lines can be colored with $f(k)$ colors in such a way that crossing lines get different colors?

J. Pach, G. Tardos, and G. Tóth, Disjointness graphs of segments, *Proc. 33rd Annual Symposium on Computational Geometry (SoCG 2017)*, to appear.

8. Imre Bárány, barany@renyi.hu

k-crossing curves in $\mathbb{R}^d$. A curve γ in $\mathbb{R}^d$ is k-crossing if every hyperplane intersects it at most k times. Thus $k \geq d$. A d-crossing curve is called *convex*.

Theorem (Bárány, Matoušek, Pór). *For every $d \geq 2$ there is $M(d)$ such that every $(d+1)$-crossing curve in $\mathbb{R}^d$ can be split into $M(d)$ convex curves.*

The proof gives $M(d) \leq 4^d$, $M(2) = 4$ and $M(3) \leq 22$.

Question: Give lower bounds for $M(d)$.

I. Bárány, J. Matoušek, A. Pór, Curves in $\mathbb{R}^d$ intersecting every hyperplane at most $d+1$ times, *J. Eur. Math. Soc. (JEMS)* **18** (2016), no. 11, 2469–2482.

9. Pavel Valtr, valtr@kam.mff.cuni.cz

Lines, line-point incidences, and crossing families in dense sets. Let P be a set of n points in $\mathbb{R}^2$ such that $\min \operatorname{dist}(P) = 1$ and $\max \operatorname{dist}(P) = O(\sqrt{m})$. Prove or disprove:

Conjecture 1. *P contains a crossing family of size $\Omega(n)$.*

Known: P contains a crossing family of size $\Omega(n^{1-\varepsilon})$.
Two lines are *essentially different* if either their direction differ by at least $1/n$, or their $\frac{1}{\sqrt{n}}$-neighborhoods do not intersect inside $\mathrm{conv}(P)$.

Conjecture 2. *P determines $\Omega(n^2)$ pairwise essentially different lines.*

Known: P determines $\Omega(n^{2-\varepsilon})$ pairwise essentially different lines.
A point p and a line ℓ determine a *rough point-line incidence* if $\mathrm{dist}(p, \ell) \le \frac{1}{\sqrt{n}}$.

Conjecture 3. *Let P as before and L a set of n pairwise essentially different lines. Then the number of rough point-line incidences is at least $\Omega(n^{4/3})$.*

10. Luis Montejano, luis@matem.unam.mx

Let X be a polyhedron. Let $\mathcal{F} = \{A_1, \ldots A_m\}$ be a polyhedral cover of X such that A_i is not empty but not necessarily connected. Let N be the nerve of $\mathcal{F}$.

Fact: Suppose that the following hold: (a) $H_1(X) = 0$, and (b) for every $i \neq j$, if $A_i \cap A_j \neq \emptyset$ then $A_i \cup A_j$ is connected. Then $H_1(N) = 0$.

Question: Suppose that the following hold: (a) $H_2(X) = 0$, (b) for every $i \neq j$, if $A_i \cap A_j \neq \emptyset$ then $A_i \cup A_j$ is connected, and (c) for every $i < j < k$, if $A_i \cap A_j \cap A_k \neq \emptyset$ then $H_1(A_i \cup A_j \cup A_k) = 0$. Is it true that $H_2(N) = 0$? The answer is yes if $m = 4$.

11. József Solymosi, solymosi@math.ubc.ca

Question 1: What is the minimum number of collinear triples in a subset of the integer grid $n \times n \times n$? If $|S| = n^{3-s}$, $S \subset n \times n \times n$, then S spans at least $\frac{n^{6-4s}}{c \log n}$ collinear triples. We (with Jozsi Balogh) do not think that this is sharp.

Question 2: Find a bipartite unit distance graph which is rigid.

12. Edgardo Roldán-Pensado, e.roldan@im.unam.mx

Centre of $BM(2)$. Let δ be the Banach-Mazur distance. Find the convex body $C \subset \mathbb{R}^2$ such that $\max\{\delta(C, D) : D \subset \mathbb{R}^2 \text{ a convex body}\}$ is minimized.

An update by Edgardo Roldán-Pensado: The answer to the problem is known. A solution appears in:

W. Stromquist, The maximum distance between two-dimensional Banach spaces. Mathematica Scandinavica (1981): 205-225.