ist,* und dass dieses Werk die Konzeption der nichteuklidischen Geometrie enthielte. Der Text enthält nur Gedanken, die die Reform und die Verbesserung der euklidischen Geometrie bezwecken.

Adresse des Editors:
Praha 10 — Zahradní město 2814.

* Dazu siehe den ersten Satz des Textes. Es ist vielleicht möglich, dass der verlorene „Antieuklid" „nach einem so ausführlichen Plane" abgefasst worden sei.

BERNARD BOLZANO

(1781 – 1848)

BICENTENARY

EARLY MATHEMATICAL WORKS

ACTA HISTORIAE RERUM NATURALIUM NEC NON TECHNICARUM

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Bolzano's Early Mathematical Achievements and Problems of His Historical Appreciation

In 1981, 200 years elapsed from the birth and more than 130 years from the death of Bernard Bolzano (Oct. 5th, 1781 in Prague - Dec. 18th, 1848 in Prague), however, the interest in his work is not only not waning but, on the contrary, it has been visibly increasing in recent decades. If we think about what drew and is drawing attention to Bolzano's personality and work in various phases of his life to this day, it is surprising how various topical groups of Bolzano's multiform activity have gradually been coming into the limelight since the first decades of the 19th century and how the groups of readers and admirers, who have interpreted the significance and meaning of Bolzano's work, have been changing. For example, during Bolzano's life the most expressive was perhaps the profound humaneness of his approach to a number of topical social problems, reflected in his opinions of human thinking, behaviour and feeling, and disseminated particularly in his preaching to the Prague undergraduates

and in some theological deliberations; however, these ideas were adopted and developed by only a relatively small group of Bolzano's admirers and pupils, as they partly depended on the ideological controversies among catholic theologists of the time. Fragments of his social opinions, demanding especially the abolishment of feudal anachronisms of state, a more profound democratism and abolishment of all forms of suppressing the Czech nation stood out under the special Czech conditions in the consciousness of the broader strata of Bolzano's centemporaries. These fragments, which in spite of their vaguenses suited the requirements declared by the representatives of the Czech national movement, associating them with the aureola of Bolzano the martyr, were frequently used ahistorically to create the Bolzano tradition which, in varying importance and form, survived until the thirties of the 20th century. The importance of Bolzano's contribution to mathematics and logics became lost in this context. This had several causes. During Bolzano's life there was in fact no mathematician or logician who understood - even at the level of the contemporaneous state of progressive scientific trends - the real content of Bolzano's ideas, to say nothing of being capable of developing them creatively further. However, in spite of this lack of understanding there was no detriment the Czech society, considered Bolzano an important mathematician and logician, who not only published important papers, but also endeavoured to apply and spread his scientific ideas in his activities. This acknowledgement, which was outwardly reflected, e. g., in Bolzano's status in the

1) The statement in the text is in no way meant to indicate that no mathematicians with a wide scope of knowledge worked in Prague in Bolzano's time. We shall speak of them later. However, they did not comprehend Bolzano's results sufficiently; but we should add that not even Bolzano's printed papers had a larger responses among his contemporaries abroad.

Royal Bohemian Society of Sciences, in no way meant that efforts would be made in Bohemia or Vienna towards a creative development of his mathematical ideas.

As has already been emphasized several times in literature,[2] the gradual "discovering" of Bolzano's heritage, particularly in mathematics, with a very few exceptions took place under conditions in which the development of mathematical cognition arrived at Bolzano's anticipated or voiced ideas, however, already at a more comprehensive and general level and mostly via other ways and others associations. That is why these parts of Bolzano's work gained (and frequently still attract) the attention of scientists in a kind of historical retrospection.[3] In spite of certain partial attention which mathematicians and exceptionally also historians of mathematics devoted to Bolzano's work since the seventies of the last century,[4] the interest in Bolzano's

2) Compare, e.g., the papers of V. Jarník from the years 1922-1962, devoted to Bolzano and published summarily in an English translation: Bolzano and the foundation of mathematical analysis, Prague, JČSMF 1981.

3) Cf., e.g., W. Dubislaw, Bolzano als Vorläufer mathematischer Logik, in: Philosophisches Jahrbuch 44, Fulda 1931, pp. 448-456; J. Berg, Bolzano's logic, Stockholm 1962; D. Johnson, Prelude to Dimension Theory: The Geometrical Investigation of Bernard Bolzano, in: Archive for History of Exact Sciences, Vol. 17, No. 3 (1977), pp. 261-295, but also, e.g., K. Rychlík, Teorie reálných čísel v rukopisné pozůstalosti Bernarda Bolzana (The theory of real numbers in the manuscript inheritance of Bernard Bolzano), in:Revue d'histoire des sciehces et leurs applications 14 (1961), pp. 313-317), etc.

4) The best known are perhaps the notes of George Cantor in the treatise Über unendliche lineare Punktmannigfaltigkeiten IV, in: Mathematische Annalen 21 (1883), pp. 51-58, 545-591, however, less known is the fact that Cantor's attention war drawn to Bolzano's papers by H. A. Schwarz in his letter of April 1, 1870 (Cf. in extenso H. Menschkowski, Probleme des Unendlichen, Braunschweig 1967, p. 228), in which Schwarz indicated that he had made use of the deliberations of Bolzano and Weierstrass in one proof Cantor

mathematical heritage began to grew from the time that the earlier anticipation of the scientific importance of his mathematical manuscripts, coincidently deposited in Vienna,[5]

had adopted. He mentioned Bolzano's Rein analytischer Beweis there and wrote directly: "Auch ich bekenne mich mit Dir zu der von Herrn Weierstrass in seinen Vorlesungen verflochtenen Meinung, dass man ohne Schlussweise, welche von Herrn W. auf Bolzanoschen Principien weiter ausgebildet ist, bei vielen Untersuchungen nicht zum Ziel gelangen könne". He called this method literarily "Bolzano-Weierstrass Schlussweise" there and probably had in mind Bolzano's theorem of the greatest lower bound of a sequence and Weierstrass' theorem that any bounded sequence has an accumulation point. Most existing literature ascribes this designation to Schwarz's paper of 1872 (Cf., e.g., Grattan-Guinness, The Development of the Foundations of Mathematical Analysis from Euler to Riemann, Cambridge, MIT 1970, p. 74). Schwarz's comment also prompts the deliberation of Weierstrass' possible direct exploitation of Bolzano's idea, or possibly also of Bolzano's other results in his Berlin lectures of the sixties.

O. Stolz attempted an evaluation of Bolzano's contribution to mathematical analysis, of course, only on the basis of Bolzano's early works (Cf.: Bolzano's Bedeutung in der Geschichte der Infinitesimalrechnung, in: Mathematische Annalen 18 (1881), pp. 255-279), when already earlier H. Hankel (Grenze, in: Allgemeine Enzyklopädie [Ersch, Gruber, eds.], Sekt., Bd. XC, Leipzig 1871) had drawn attention to Bolzano's concept of the limit, etc.

R. Dedekind mentioned Bolzano's Paradoxien des Unendlichen in the preface to the 2nd edition of his book Was sind und was sollen die Zahlen (1893), where he also pointed out that this treatise was quite unknown to him at the time he was finishing the first edition of his work (1887), which also applies to Cantor's papers in Crelle Journal (1878). However, thus is in considerable contradiction to his very lively correspondence with Cantor, which hat already begun in 1872 (Meschkowski, op. cit., p. 26 ff.), oriented towards, among other problems, the problem of sets an infinite number of elements.

5) Bolzano, who sought a continuator of his mathematical work, particularly as his doubts whether he would be able to finish the comprehensive presentation of mathematics according to his ideas, grew, towards the end of his life invested his hopes in young Robert Zimmermann (1824-1898) to whom he then willed his mathematical manuscripts. Of course, Robert Zimmermann concentrated only on philosophy,

was proved and as soon as the Czech mathematicians began to study this heritage. Thus, since the twenties, especially due to M. Jašek,[6] to the analysis of Bolzano's theory of

and later became associate professor (1849) and professor of philosophy (in Prage since 1852, in Vienna since 1861) and member of the Viennese Academy of Sciences. Bolzano's mathematical inheritance he then handed over to the Viennese Academy of Sciences in 1882 and it passed it on, while Zimmermann was still alive, to the manuscript department of the present Österreichische Nationalbibliothek.

As regards Bolzano's appreciation of Robert Zimmermann, we can find sufficient evidence in his correspondence with M. J. Fesl (published in abbreviated form in: Wissenschaft und Religion in Vormärz. Der Briefwechsel Bernard Bolzanos mit Michael Josef Fesl 1822-1848, Berlin 1965). In 1843 he was enthusiastic about him: "Die meiste Freude macht mir der junge Zimmermann, dessen Talente und vortreffliches Herz seit jedem Tag herrlich entfaltet ... aus der höheren Mathematik sich in neun Monaten mehr Kenntnisse angeeignet, als mancher andere wohl in drei Jahren vermöcht hätte ... Er ist ein herrlicher Junge ...". But later he wrote Fesl "dass R. Zimmermann zwar die vortrefflichsten Talente, aber sehr wenig Fleiss und Ausdauer besitzt ... Ich fürchte, ... er wird zur Ausführung meiner Zwecke nicht taugen!"(Cf. also Introduction in: Eduard Winter, Robert Zimmermanns philosophische Propädeutik und die Vorlagen aus der Wissenschaftslehre Bernard Bolzanos, Wien 1975, pp. 15 f.).

6) M. Jašek (1879-1945), secondary school teacher in Pilsen (West Bohemia), already in 1921 drew attention to some results contained in Bolzano's mathematical manuscripts. Compare with M. Jašek, Aus dem handschriftlichen Nachlass Bernard Bolzanos. Das erste, historisch nachweisbare Beispiel einer stetigen nirgends differenzierbaren Funktion. Enthalten in dem Manuskripte Bolzanos "Functionenlehre" (Nationalbibliothek in Wien), in: Věstník Královské české společnosti nauk, Třída matematicko-přírodovědecká (1921), č. 1, pp. 1-32. Cf. also: Jahresbericht der deutschen Mathematiker-Vereinigung 31 (1922), 2. Abt., pp. 109-110 and Časopis pro pěstování matematiky a fysiky 51 (1922), pp. 69-76, and 53 (1923-4), pp. 102-109. Since then, with the support of the Royal Czech Society of Sciences Jašek studied Bolzano's mathematical manuscripts and considerable merit is due to him for preparing the edition of Bolzano's treatise, published by this Society.

functions carried out by V. Jarník,[7] to the later studies of K. Rychlík[8] oriented towards Bolzano's theory of numbers, logics and the theory of real numbers, and to other works, the mathematical and historical community became acquainted with some of Bolzano's selected results. This trend continued also in the subsequent period.[9] The overall increase of interest in Bolzano's work, after various attempts to publish

7) V. Jarník, O funkci Bolzanově (On Bolzano's function), in: Časopis pro pěstování matematiky a fysiky 51 (1922, pp. 248-264, compare with the English translation in Jarník's publication referred to in quot. 2, pp. 67-81. Idem: Bolzanova Functionenlehre (Bolzano's Functionenlehre), in: Časopis pro pěstování matematiky a fysiky 60 (1931), pp. 240-265, in English in the reference publication on pp. 43-66.

8) K. Rychlík (1885-1968), professor of mathematics on the Czech Technical University in Prague, began to study Bolzano's mathematical manuscripts early (compare K. Rychlík, Über eine Funktion aus dem Bolzanos handschriftlichen Nachlasse, in: Věstník Královské české společnosti nauk (1922), vol. IV, pp. 1-20; also, e.g., K. Rychlík, La théorie des fonctions de Bolzano, in: Atti del Congresso Internationale dei Matematici, Bologna 3.-10. Settemre 1928, (VI), 6, Bologna 1932, pp. 503-505.); later he studied Bolzano's work more systematically. Apart from the edition of two volumes in Spisy B. Bolzana (The Works of B. Bolzano), published by the Royal Bohemian Society of Sciences (Vol. 1, Functionenlehre, Praha 1930; Vol. 2, Zahlentheorie, Praha 1931) he also published Theorie der reellen Zahlen im Bolzanos handschriftlichen Nachlasse (Praha 1962) and a whole series of other treatise, e.g., Úvahy z logiky v Bolzanově rukopisné pozůstalosti (Deliberations in logics in Bolzano's manuscript inheritance), in: Časopis pro pěstování matematiky a fysiky 83 (1958), pp. 230-235 (also in: Czechoslovak Mathematical Journal 8 (1958), pp. 197-202); La théorie des nombres réels dans un ouvrage posthume manuscrit de Bernard Bolzano, in: Revue d'histoire des sciences et de leurs applications 14 (1961), pp. 313-327; Sur les contacts personels de Cauchy et de Bolzano, ibid. 15 (1962), pp. 163-4.

9) Later, e.g., V. Jarník summarized his research in the treatise Bernard Bolzano a základy matematické analýzy (Bernard Bolzano and the foundations of mathematical

his scientific work,[10] eventually led to a grandiously

analysis), in: Zdeňku Nejedlému Československá akademie věd, Praha 1953, pp. 450-458, in English in V. Jarník's paper referred to in quot. 2, pp. 33-42.

10) Very shortly after Bolzano's death M. J. Fesl attempted to publish Bolzano's whole work. His project, divided into 20 volumes and subdivided into five main groups, however, did not mature and the means accumulated for this purpose were only sufficient to publish four volumes of Bolzano's Erbauungsreden (Praha 1849-1852); however, not even Braumüller's Viennese edition (Dr. B. Bolzano's gesammelte Schriften. Neue Ausgabe in zwölf Bände, Wien 1882) was successful, because it concentrated primarily on publishing Bolzano's treatise which had already been published and omitted the most important part of his mathematical heritage.

In 1931 the Royal Bohemian Society of Sciences in Prague began to publish Spisy Bernarda Bolzana (The Works of Bernard Bolzano), concentrating on the unpublished, particularly the mathematical inheritance; however, only five volumes were published:

Vol. 1, Functionenlehre (K. Rychlík, ed.), Praha 1930
Vol. 2, Zahlentheorie (K. Rychlík, ed.), Praha 1931
Vol. 3, Von dem besten Staate (G. Kowalewski, ed.), Praha 1932
Vol. 4, Der Briefwechsel B. Bolzano's mit F. Exner (E. Winter, ed.), Praha 1935
Vol. 5, Memoires géométriques (J. Vojtěch, ed.), Praha 1948.

The result of the further study of Bolzano's mathematical inheritance, stimulated by the Czechoslovak Academy of Sciences, was Rychlík's interpretation of Bolzano's theory of real numbers, published in 1962 (cf. quot. 8).

Since 1961 the Czechoslovak Academy of Sciences has been preparing an edition of Bolzano's mathematical manuscripts, deposited in Vienna. Within the scope of this preparation Dr. Kazimír Večerka has rewritten these manuscripts in their various versions, has started to compare them and to edit them. In connection with the preparation, also Bolzano's fragment Anti-Euklid (K. Večerka, ed.) was published, in: Sborník pro dějiny přírodních věd a techniky, Vol. 11 (1966), pp. 203-216, and some scientists had the possibility of drawing on these transscriptions (e.g., L. Nový, Origins of Modern Algebra, Praha 1973, pp. 90 - 92). Earlier than the tedious processing of the extensive manuscripts, the purpose of which, according to current edition principles, was to summarize in a single edition

conceived editorial attempt of the Gesamtausgabe;[11] in the field of mathematics, the edition is so far concentrated on the gradual publishing of his manuscripts and this has still a long way to go. In an effort to facilitate the researchers' study of Bolzano's mathematical heritage, the Institute of Czechoslovak and General History of the Czechoslovak Academy of Sciences (Department of History of Natural Sciences and Technology) in co-operation with the organizing committee of the conference "Impact of Bolzano's Epoch on the Development of Science" (Prague, Sept. 8th - 12th, 1981) is publishing in this volume, at the occasion of the 200th anniversary of Bernard Bolzano's birth, a reprint of five of Bolzano's mathematical papers, originally published in the years 1804 - 1817. The following papers are involved:

1. Betrachtungen über einige Gegenstände der Elementargeometrie (a reprint of the original issue published by K. Barth in Prague in 1804, (10)+ 62 pp.)
2. Beyträge zu einer begründeteren Darstellung der Mathematik (a reprint of the original issue published by C. Widtmann in Prague in 1810, XVI + 152 pp.)
3. Der binomische Lehrsatz und als Folgerung aus ihm der polynomische, und die Reihen, die zur Berechnung der Logarithmen und Exponential Grössen dienen, genauer als bisher erwiesen (reprint of the original issue published by C. W. Enders in Prague in 1816, XVI + 144 pp.)
4. Rein analytischer Beweis des Lehrsatzes, dass zwischen je zwey Werthen, die ein entgegensetztes Resultat gewähren, wenigstens einer reelle Wurzel der Gleichung liege (a reprint of the original issue, published for the Royal Bohemian Society of Sciences by G. Hass in Prague in 1817, 60pp.)
5. Die drey Probleme der Rectification, der Complanation und der Cubirung ohne Betrachtung des unendlich Kleinen, ohne die Annahme des Archimedes und ohne irgend eine nicht streng erweisliche Voraussetzung gelöst; angleich als Probe einer ganzlichen Umstaltung der Raumwissenschaft allen Mathematikern zur Prüfung vorgelegt (a reprint of the original issue published by P. G. Kummer in Leipzig in 1817, XXIV + 74 pp.).

The reprint retains the original pagination, given mostly at the upper margine of the page and, for the sake of orientation of the reader of this edition, an auxiliary pagination is given in the side margine of the page.

Among Bolzano's mathematical works, concluded in Bolzano's early period (1815), there also belongs Versuch einer objekti-

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the various preserved versions, Bernard Bolzano's Gesamtausgabe, governed by simpler edition principles, began to be published (as regards the problems associated with these principles with a view to further historical work, compare, e. g., with the review of Volumes II A 7 and 8, in: Dějiny věd a techniky 10 (ů977), pp. 245-247).

11) Bernard Bolzano - Gesamtausgabe has been published since 1969 in the Fromann-Holzboog Publishing House in Bad Cannstatt near Stuttgart and is divided into five groups. The "introductory" group (denoted E) contains, apart from the biographical volume also the bibliography and catalogues of the manuscript inheritance. The next group (I) is devoted to the re-edition of treatise published earlier. Group (II), in which Bolzano's manuscript inheritance is concentrated, is divided into treatise from the inheritance (A) and scientific diaries (B). Group (III) will encompass Bolzano's correspodence in five volumes and group (IV) is oriented towards the edition of documents from the life and work of B. Bolzano.

By 1980 of the anticipated about 60 volumes of the edition only 11 had been published, containing apart from group E particularly groups II A and II B. The problem of the edition is its criticalness, because the filiation of various versions of Bolzano's manuscripts in the published volumes was not taken into account.

ven Begründung der Lehre von den drei Dimensionen des Raumes, which was, however, published in Prague as late as in 1845 (Abhandlungen der Königlichen böhmischen Gesellschaft der Wissenschaften, 5. Folge, Bd. 4). Its was not included in this volume for this reason.[12)]

For the convenience of researchers thus these reprints give Bolzano's principal mathematical works, published in the period of his early research activity, even before the principal turning point in his life, caused by his theological arguments with the contemporaneous, particularly Viennese church hierarchy. These arguments re-oriented Bolzano's attention for a whole long period of his life, fortunately without distracting him from his creative work in mathematics, but quite certainly preventing him from publishing his own results, many of which has already been, judging by various preserved versions of manuscripts, prepared for the press, quite against Bolzano's original intentions.[13)] After publishing these early works, when he wanted his results to undergo public criticism even before he started to think of elaborating the whole system of mathematics,[14)] he rejected this intention. As if he had verified the power of his own mathematical invention without any response, at the end of the twenties of the 19th century Bolzano began to fulfil his concept of mathematics, part of which should have been Wissenschaftslehre,

12) The paper was included in Vol. 5 in quot. 10 of the referenced edition of Bolzano's works (Praha 1948, pp. 51 - 67).

13) Bolzano thus discarded his point of view, voiced in Rein analytischer Beweis (pp. 26 f.) in which he claims that he had decided "in keiner Wissenschaft je mit der Herausgabe eines vollständigen Lehrbuchs anzufangen, sondern in jeder meine von den gewöhnlichen abweichende Begriffe nur erst in einzelnen Abhandlungen bekannt zu machen," already in 1804.

14) Ibid.

Grössenlehre and Raumlehre. Apart from an extensive systematic presentation of the methods of building-up of science, made in Wissenschaftslehre,[15)] essentially concluded around 1830, and the Paradoxien des Unendlichen[16)] prepared just before his death, his whole project remained in manuscript fragments. The more important are Bolzano's early mathematical works for the complex study of Bolzano as a mathematicians, as well as for including Bolzano's mathematics in the endeavours of the time.

We have already mentioned the attention which was devoted to Bolzano's work by the mathematicians and historians of mathematics in analysing its individual parts. Nevertheless, his overall historical appreciation will require a very demanding research effort, the results of which does not depend just on the extent of the published mathematical manuscripts. Let us attempt, particularly in connection with reprinting Bolzano's early mathematical works, to outline the main difficulties encountered by the historical evaluation of Bolzano's contribution.

In historical research of scientific cognition, it is important to analyse the individual past results (discoveries) at least from two different points of view. First of all, in evaluating the significance of these results we cannot manage without their evaluation from the point of view of the contemporaneous tendencies in the development of science. Although there is a great danger of actualizing the knowledge of the past particularly in mathematics, i. e. translating them into the present language of science which frequently leads not only to a distortion of the result, but also of the

15) Wissenschaftslehre. Versuch einer ausführlichen und grösstentheils neuen Darstellung der Logik mit steter Rücksicht auf deren bisherige Bearbeiter, Bd. I-IV, Sulzbach 1827.

16) Published by Příhonský in Leipzig in 1851.

contemporaneous understanding and approach to the problem, this danger is multiply outweighed by the advantages of understanding the more profound meaning of the studied moment of evolution of mathematics. We are of the opinion - and the study of Bolzano's mathematics has proved it - that a gradual accumulation of these analyses from the aspect of the hundred years of evolution of modern mathematics, in spite of all peripeteia and the still uninvestigated (or unpublished) parts of Bolzano's heritage, will enable a relatively just comprehension of the mathematical content of his work. We can go one step further and voice the hypothesis that a further study of Bolzano's mathematical work from the indicated point of view, will hardly yield any surprising results. The more important is the second aspect, very necessary in historical analysis, i. e. to understand the location of the work and his creator in connection with the time on which it was being created, developed and applied. In spite of a number of important historical studies, the understanding of Bolzano's mathematical work in the historical associations of the 1st half of the 19th century ist still at its very beginning.[17]

Bolzano's mathematical work is in many respects atypical with regard to the principal trends of mathematical research of its time. In the forming of Bolzano's mathematical endeavour,[18] several important determining factors were involved, which then also affected the form of and response to Bolzano's results. If we disregard Bolzano's exceptional talent, which, we may perhaps say, predisposed him to work in mathematics and was reflected in Bolzano's exceptional capability and power to effect logically accurate abstract deliberations and to develop logical constructions, there

17) Cf. J. Folta, Life and scientific endeavour of Bernard Bolzano, introductory study of Jarník's edition (op.cit. quot. 2, pp. 11-31).

18) His own mathematical notes, called "Miscellanea mathematica" in his inheritance and deposited in 24 volumes in

were various factors, which limited substantially the development and exploitation of Bolzano's mathematical talent, and which acted frequently against this talent. First of all, Bolzano himself devoted most of his intellectual powers to other problems, which attracted him more for various reasons. Perhaps we may exagerate a little by saying that the interest in the "welfare of mankind" diverted Bolzano's concentration from mathematical work, or that mathematics was just one of the regions, of a whole series which he dwelled in his endeavour for moral improvement of mankind. If we add the external conditions, independent of Bolzano's will (at first work duties and poor health, later the strong influence of the small group of his disciples, etc.), we find that Bolzano only devoted a minor part of his life to the study of mathematics. Essentially two periods are involved, which are not even firmly time determined and in which his study of mathematical problems also yielded positive results. In the first place, these were the first two decades of the 19th century, when Bolzano created his clear-cur concept of mathematics and of the possibilities and aims of its consistens logical construction. Although he tried to formulate these ideas of the necessary restructuring of mathematics generally in his paper "Beyträge zu einer begründeteren Darstellung der Mathematik",[19] he himself understood the importance of his own

the Vienna part of the hereditaments, draw a plastic picture of Bolzano's study of mathematics, together with the retrospectively styled facts in his autobiography, written for Anna Hoffmann (compare Lebensbeschreibung des Dr. B. Bolzano M. J. Fesl , Sulzbach 1836). The first volumes of Miscelanea mathematica, containing notes up to 1811 were published in Gesamtausgabe II B 2, Teil 1, 2 (B. v. Rootselaar, and Anna van der Lugt, eds.).

19) In this edition pp. 83-250. In the original edition it is mentioned that "erste Lieferung" is involved; the continuation remained in manuscript form and was published in Gesamtausgabe Vol. II A 5.

attempts to execute his reforming proposals in partial experiments. With this immediately in mind, he concieved particularly three papers, published in the years 1816-17, i. e. "Der binomische Lehrsatz ...",[20] "Rein analytischer Beweis ..."[21] and "Die drey Probleme ..."[22]. To these three papers, in which Bolzano prudently selected then important problems, belonging to contemporaneous mathematics, to discuss, we can add also Bolzano's mathematical first-fruit already disclosing the author's talent, i. e. "Betrachtungen über einige Gegenstände der Elementargeometrie".[23]

The second, relatively equally long period of Bolzano's mathematical work, included the thirties and forties. In between Bolzano devoted considerable attention to elaborating his extensive work Wissenschaftslehre,[24] containing not only

20) In this edition pp. 254-412.

21) In this edition pp. 417-476.

22) In this edition pp. 479-590.

23) In this edition pp. 1-80. B. Bolzano inscribed these first-fruits to professor of elementary mathematics Stanislav Vydra and wrote verbatim: " ... zum Beweise einer unbegränzten Hochachtung und Dankbarkeit gewidmet von seinem ehemaligen Schüler dem Verfasser". On Vydra refer to quot. 31.

24) Cf. quot. 15; since 1829 Bolzano and his friends sought the possibility of publishing this extensive work, but Bolzano continued working on it, which also caused difficulties in its printing. As regards the complicatedness of the effort to publish the work, refer to Karel Berka, Blanka Prokešová, Bolzanovy boje o vydání a uznání Vědosloví (Bolzano's struggle for publishing and acknowledgement of "Wissenschaftslehre"), in: Filozofický časopis 27 (1979), pp. 697-725. In the introduction to this treatise the authors point out the association between Beyträge zu einer begründeteren Darstellung der Mathematik (published in 1810) and Bolzano's deliberations and preparations of the future Wissenschaftslehre. Considerable evidence of this is contained in Vol. II A 5 of the Gesamtausgabe and in Miscellanea mathematica. However, historically the relations of contents between Bolzano's mathematical efforts in the first decade of the 19th century

some mathematical stimuli, but also, and this is particularly important, developing and advocating the idea of a comprehensive scientific interpretation of scientific disciplines. This should have become an extensive mathematical work, slightly inaccurately called Grössenlehre, on which Bolzano worked quite earnestly with intermissions, but which, as he was sorry to say, he did not finish, nor could he finish according to his intentions, and parts of which remained in manuscript form.[25] Thus, the main published result of this period were his posthumously published Paradoxien des Unendlichen, the importance of which was relatively soon pointed out by G. Cantor,[26] and which have been repeatedly published and translated since.

Another important factor, which affected the development of Bolzano's work substantially, was his approach to mathematical problems. Two tendencies met here. On the one hand,

and his logic have been investigated only very little.

25) As early as on July 24th, 1842 Bolzano wrote Fesl: Was aber das grössere Werk über die Mathematik anlangt (ein eigentliches Lehrbuch sollte es wohl nie werden), daran ich vor zehn Jahren und mehr gearbeitet habe, so schwindet mit jedem Tage mehr in Hoffnung, dass ich je es selbst werde zustande bringen können; ja sie ist eigentlich schon vor Jahr und Tag so gut als aufgehoben worden. Und ich seufze nur nach einem jungen Gelehrten, der sich, solange ich noch lebe, unter meiner Anleitung mit meinen Begriffen vertraut macht und dann die Ausarbeitung des Werkes aus sich nimmt ... (compare with Wissenschafts und Religion in Vormärz, der Briefwechsel B. Bolzanos mit M. J. Fesl, 1822-1848, Berlin 1965, p. 318).
A little later (ibid., p. 324, in the letter of February 1843) having recovered from an illness which, as he was convinced, had brought him close to death, in a deliberation, how to make the best of the rest of his life, he only reckened with finishing a few shorter treatise. Bolzano's opinions were permanently influenced by the death of Anna Hoffmann (Apr. 20th, 1842).

26) Cf. quot. 4.

Bolzano was an autodidact in mathematics, for whom mathematical deliberations represented a field in which he could exploit his logically thinking mind, and this enabled him to arrive at various improvements or attempts at new views in some aspects relatively quickly and with the little knowledge with which contemporaneous university mathematics could provide him.[27] However, this autodidaction of Bolzano's also displayed another tendency, affecting the development at the time, i. e. that there was not then such a live scientific medium in Bohemia which would have been capable, as we have already pointed out, of understanding Bolzano's efforts at an adequate level, or of forming fruitful simulti or tasks for his mathematical work. This does not mean that mathematicians did not work scientifically in Bohemia in the 1st half of the 19 century; on the contrary, Bolzano's teacher was František Gerstner (1758-1832), who had a relatively wide overview of mathematics and mechanics of the 18th century and who was able to appreciate Bolzano's mathematical talent.[28] Of course,

27) As we can judge from the contemporaneous university teaching of mathematics in Prague (compare with L. Nový, Matematika na pražské universitě v druhé polovině 18. století (Mathematics at the Prague university in the second half of the 18th century), in: Acta Universitatis Carolinae, Historia Universitatis, Tom. II., Fasc. 1, pp. 35-57), Bolzano did not undergo more systematic mathematical training at the university not with his two teachers, S. Vydra and F. Gerstner. It corresponded to his statement in his autobiography of how Kästner's textbook (i. e. Anfangsgründe der Mathematik, 1st ed. 1758-69; at the end of the 18th century the third edition was published in Göttingen, 1774-1799, in 12 volumes, some volumes, however, also in subsequent editions) stimulated him in studying mathematics. Also his records in Miscellanea mathematica are evidence of Bolzano's independent way to mathematics (compare with Gesamtausgabe II B 2, Teil 1,2).

28) Franz Josef Gerstner (1756-1832) studied in Prague and, after working briefly at the astronomical observatory in Vienna, he first substituted and then in the years 1787 - 1822 worked as professor of higher mathematics at the university in Prague. He was well-known for his astronom-

his own scientific endeavours, originally devoted to celestial mechanics, were gradually overlapped by his effort to exploit mathematics and mechanics practically in technology. In this respect, he also oriented his double function as the only professor of higher mathematics at the Prague university (up to 1821), and as the initiator and later director of the reorganized (1806) Prague technical university.[29] With a view

ical work. His scientific authority was considerable and he was appreciated as an organizer. B. Bolzano wrote a treatise about Gerstner and published it in 1837 in the Abhandlungen der Königlichen böhmischen Gesellschaft der Wissenschaften (Leben Franz Joseph Ritters von Gerstner).

Literature makes repeated mention of Gerstner highly appreciating Bolzano's mathematical talent, and he was sure to have supported Bolzano to become head of the Department of elementary mathematics, the position having become vacant due to the death of S. Vydra in 1804. As we know, Bolzano was also interested in the newly established chair of religious sciences. It is now difficult to bring evidence why Bolzano finally was awarded the latter chair; possibly Bolzano's subjective inclination to the chair of religion was motivated by his own decision to become a priest. Literature, however, sometimes omits the fact (compare E. Winter, Bernard Bolzano und sein Kreis, s. II, in: Bernard Bolzano. Ein Denker und Erzieher im Österreichischen Vormärz. Wien 1967) that the chair of elementary mathematics was warded, quite naturally and in accordance with custom, to L. J. Jandera, who was older and had successfully substituted in lectures for the blinded Vydra. However, the attitude of F. Gerstner after 1820 is interesting. E. Winter pointed out (Der Bolzanoprocess. Dokumente zur Geschichte der Prager Karlsuniversität im Vormärz. Brünn - München - Wien 1944, p. 41) that after his removal Bolzano requested the position of adjunct of professor of higher mathematics F. Gerstner (ibid., pp. 159-160). Gerstner supported this application and also explained why after 1804 he gave preference to Jandera in awarding the chair of elementary mathematics. Compare L. Nový, K otázce Bolzanovy profesury matematiky v r. 1821 (On the question of Bolzano's professorship of mathematics in the year 1821), in: Zprávy Komise pro dějiny přírodních lékařských a technických věd ČSAV, No. 7, Praha 1961, pp. 28-30.

29) As regards the role of Gerstner in reorganizing the Prague technical university, refer to Dějiny Českého vysokého uče-

to the poor technical development of industry in Bohemia - which already was the highest in the Austrian monarchy - at the end of the 18th and beginning of the 19th century, the call for mathematical knowledge was little, and Gerstner adapted his own work, in which practical questions predominated more and more, accordingly.[30] On the contrary, Bolzano was in no way associated with these efforts of Gerstner's, they were alien to his ego and he did not understand them. The practical associations of mathematics, connected with the contemporaneous development of physics were not acknowledged or understood well by Bolzano. Bolzano did not see the essence of exploiting mathematics and its contemporaneous tasks, but he unilaterally emphasized the necessity for a uniform, logically elaborated interpretation of mathematics as a whole, which then ensued from his more general philosophical and theoretical ideas and which was applied to his work on the Grössenlehre in the second period of his mathematical creative work.

As regards Czech mathematicians we have only mentioned Gerstner. Also others worked in Bohemia: after S. Vydra [31]

ní technického, Part 1, Vol. 1, Praha 1973, particularly p. 152 ff., where detailed references are also given; Gerstner's original proposal of 1798 required that the Prague Technical University be restructured according to the model of the Ecole polytechnique in Paris, recently established.

30) The expression of Gerstner's interests is the three-volume textbook "Handbuch der Mechanik" (Praha 1831-1834), his support for constructing railways which was also continued by his son, František Antonín Gerstner, and, finally, his importance is also reflected in a number treatise; ref.to Dějiny exaktních věd v českých zemích do konce 19. století (History of Science in the Bohemian Lands up to the end of the 19th century), Praha 1961, particularly pp. 175-177.

31) Stanislav Vydra (1741-1804), professor of elementary mathematics at the Philosophical Faculty of the Prague Universisty in the years 1771-1803. More detailed material concerning the life and work of S. Vydra can be found in the

the Department of Elementary Mathematics was headed by L. J. Jandera,[32] the astronomer A. M. David worked in Prague, [33] at the technical university A. Bittner,[34] Chr. Doppler, [35]

treatise of Miloslav Fuka, Materiály k dílu Stanislava Vydry, českého matematika z konce 18. století (Materials concerning the work of Stanislav Vydra, Czech mathematicians of the end of the 18th century), in: Sborník pro dějiny přírodních věd a techniky, Vol. 3, Praha 1957, pp. 179 -195. Among other things, he also gives a commented list of Vydra papers. In appreciating Vydra's mathematical work the author came to the conclusion that Vydra's "relatively numerous mathematical treatise are of an elementary nature and usually dependent on the current textbooks of the time" (ibid., p. 190). However, he did appreciate Vydra for his indubitable pedagogical merits and also because he as one the first mathematicians avowed himself to Czech nation and wrote a textbook of arithmetic in Czech.

32) Ladislav Josef Jandera (1776-1857) was professor of elementary mathematics at the Philosophical Faculty of the Prague University in the years 1803-1857. He was a pedagogue in the first place. It is worth mentioning his paper "Beiträge zu ... gründlicheren Behandlung ... der Arithmetic" (Praha 1840; he published a treatise with similar opinions already in 1812) in which he attempted to explain the so-called "mathematical method".

33) Alois Martin David (1757-1834) worked at the Prague astronomical observatory since the eighties; he was director of the observatory since 1799. As an astronomer he mainly dealt with observation. It is worth meintioning Abel's notes, who passed through Prague on his way to Italy in 1826 and met with prof. David and was informed of F. Gerstner, but knew nothing of Bolzano. Ref. to Abel's letters in C.A.Bjerknes' work, Niels Henrik Abel, Berlin 1930, p. 72.

34) Adam Bittner (1777-1844) worked in Prague first at the astronomical observatory (1800) and then (1802) as professor of mathematics at the Prague Polytechnique. In 1837 he became director of the observatory and professor of astronomy at the Prague University.

35) Christian Doppler (1803-1854), after studying and working as a pedagogue in Vienna, came to Prague in 1835, where he became professor of mathematics at the Prague Polytechnique in 1841. In 1847 as professor he moved to Banská Štiavnica and since 1849 he worked as professor in Vienna. When he was in Prague he got acquainted with

later young F. X, Moth[36] and others.[37] In 1826 the Depart-

B. Bolzano who held him in high esteem. He published several mathematical papers while in Prague (e.g., Versuch einer analytischen Behandlung beliebig begrenzter und zusammengesetzter Linien, Flächen und Körper, nebst einer Anwendung davon auf verschiedene Probleme der Geometrie, Descriptive und Perspective, Praha 1839), of course, his main result was the paper Über das farbige Licht der Doppelsterne, published in 1842 in the Abhandlungen der königlichen böhmischen Gesellschaft der Wissenschaften in Prag (5. Folge, 2. Bd), containing the so-called Doppler effect. It was indeed Doppler's theoretical thinking, running into very complicated and abstract deliberations even in physics, which Bolzano highly appreciated. Doppler's work became the subject of two Bolzano's articles (Ein Paar Bemerkungen über die neue Theorie in... Doppler's Schrift Über das farbige Licht der Doppelsterne ..., in: Annalen der Physik und Chemie 60, Leipzig 1843, pp. 83-88, and C. Doppler's neueste Leistungen auf dem Gebiete der physikalischen Apparatenlehrakustik, Optik und optischen Astronomie, in: Annalen der Physik und Chemie, Leipzig 1847, pp. 530-535.

36) Franz Xaver Moth (1802-1879) studied in Prague and at the beginning of the twenties substituted for Gerstner at lectures in higher mathematics. In the years 1849-1879 he was professor of mathematics at the university in Vienna. He dealt in analysis. In 1827 he published a more extensive work, Theorie der Differenzial-Rechnung (Praha, 260pp). It is worth mentioning that Moth referred very ardently to B. Bolzano, whom he considered to be a gem of the Prague University, loved by all pupils, in the biography he presented to the Academy of Sciences in Vienna in 1849 in connection with being elected corresponding member. He claimed that "hatte ich später das Glück, in den engern Kreis der Wenigen zu gehören, die er in seine näheren Umfang zog"; after Bolzano was deprived of the professorship, he said, he kept in secret contact with Bolzano at Bolzano's own suggestion. Bolzano's manuscripts contain an outline (from 1827) of a review of Moth's referenced book, in which Bolzano appreciates the hopes provided by Moth's youth, diligence and talent, but clearly does not agree with his opinions. Bolzano's review of Moth's "Theory of the differential calculus..." was published in the Monatsschrift der Gesellschaft des vaterländischen Museums in Böhmen 1, Prag 1827, pp, 79-82. For details of these questions refer to L. Nový, Bolzano a Moth (Bolzano and Moth), in: Zprávy Čs. společnosti pro dějiny věd a techniky 11, 1969, pp. 52-56. However, in the extensive edition of Bolzano's correspondence with Fesl (Cf. quot. 25)

ment of Higher Mathematics at the Prague university was taken over by J. P. Kulik who, apart from the solution of algebraic equations also devoted a considerable effort to compiling a table of prime numbers up to 100 000 000.[38] Bolzano remainded alone among all these Czech scientists, although they acknowledged him and undoubtedly some of them tried to communicate with him scientifically [39] or even work on problems closely related to his.[40] However, Bolzano's attempts to influence

Moth is not mentioned even in the index.

37) For example, the younger Karl Jelínek (1822-1876) of whom Bolzano wrote to Fesl in 1848, the correspondence mentioned in the previous quot., not particularly laudably (pp. 409-410); Josef Havle (1763-1840) and Karl Wiesenfeld (1802-1870), teaching, among other subjects, descriptive geometry at the Prague technical university. Of course, Bolzano was in direct contact with other mathematicians, e.g., with his pupil Antonín Slivka of Slivice and in a certain way also with Wilhelm Matzka (1798-1891), for whom he prepared acceptance in the Königlich böhmischen Gesellschaft der Wissenschaften and his coming to Prague in general with his reviews in the Abhandlungen of this scientific society. Compare with the referenced correspondence Bolzano - Fesl, which is indicative of the background of these contacts, as well as of Bolzano's opinions of Matzka's mathematical papers.

38) Jakub Filip Kulik (1793-1863) came to Prague in 1826 as professor of high mathematics, i.e. to the chair vacant by Gerstner, and worked there until his death. He dealt particularly in the theory of numbers, numerical methods, etc. His tables of divisors are renowned. Ref. to L. Nový, O Kulikových tabulkách dělitelů (On Kulik's tables of divisors), in: Sborník pro dějiny přírodních věd a techniky 8 (1963), pp. 43-52, also I. Ya. Depman, Zamechatelnye slavyanski vychisliteli G. Vega i Ya. F. Kulik (Important Slavonic mathematicians G. Vega and J. F. Kulik), in: Istoriko-matematicheskie issledovaniya, Vol. VI, Moscow 1953, pp. 573-608.

39) Apart from Robert Zimmermann this also applied particularly to Christian Doppler and Antonín Slivka of Slivice; Bolzano considered these three to have opinions of mathematics close to his own.

40) The scientific publishing activity of the contemporaneous Prague mathematicians was by no means particularly ex-

the mathematical community in Bohemia remained useless.[41]

tensive (for a brief picture refer to Dějiny exaktních věd v českých zemích do konce 19. stol., op.cit. quot. 30, pp. 133-160), however a number of them, in a certain contact with Bolzano, attempted to present a justified interpretation of the foundations of mathematics, expecially of the differential calculus: Moth, Bittner, Buquoy, Kulik, Jandera, Doppler. Cf. L. Nový, Základy matematické analýzy u Bolzanových pražských současníků (Foundations of mathematical analysis with Bolzano's Prague contemporaries), in: Sborník pro dějiny přírodních věd a techniky 6, Praha 1961, pp. 28-43.

41) Particularly in the forties Bolzano took active part in the work of the Royal Bohemian Society of Sciences and was its official, also its director (1842-43). Under his influence an independent section of mathematics, of which he was chairman, was established. In the years 1844-45, for example, he lectured there on the topic Lösung verschiedener in den mathematischen Wissenschaften vorkommenden Paradoxien (ref. to Sectionsberichte der königlichen böhmischen Gesellschaft der Wissenschaften 1845, p. 8 and the Abhandlungen of the same society, 5. Folge, Bd. 4.). As he himself mentioned in his letter to Fesl (of Jan. 17th, 1845, ref. to the correspondence of Bolzano - Fesl, p. 345, referenced above) he intended to write the treatise: Über verschiedene in den mathematischen Wissenschaften vorkommende Paradoxien, i.e. the future Paradoxien des Unendlichen.

Bolzano hat lectured in the Society already earlier. In his letter of July 24th, 1842 to Fesl concerning the origin of his lectures on the mathematical method in the Royal Bohemian Society of Sciences, he wrote: Wenn niemand anderer etwas vortzutragen wusste, so musste ich als der Geschäftsleiter herhalten und erbot mich, etwas über den "Ideengang in meinem System der Mathematik" nur aus dem Stegreife zu schwatzen. (Bolzano - Fesl correspondence, cf. quot. 25, pp. 318-319, where data on the publishing of the contents of these lectures in the Sitzungsberichte and Abhandlungen of the Society are also given).

As regards the relation of Bolzano to Czech scientific medium, refer to J. Levora, Bernard Bolzano a Královská česká společnost nauk (Bernard Bolzano and the Royal Bohemian Society of Sciences), in: Akademiku Václavu Vojtíškovi k 75. narozeninám, Praha 1958, pp. 69-89; I. Seidlerová, Ještě několik poznámek o Bolzanově vztahu k českému vědeckému prostředí (A few more comments on Bolzano's relation to the Czech scientific medium), in:

Bolzano's knowledge of contemporaneous mathematics developed quickly and already during the first period they became quite extensive.[42] There was hardly one more important paper in mathematics published without Bolzano knowing of it with relatively little delay, although in some cases he may have got to know of it indirectly or he did not have the time or interest to actually study it.[43] In other words, Bolzano's unilateral understanding of mathematics is not just due to the actual isolatedness from world scientific trends. On the contrary, Bolzano himself considered the whole creative effort of the contemporanesous mathematical and physical science from his own specific point of view, which emphasized the highly accurate interpretation of existing knowledge rather than obtaining new mathematical knowledge, required to solve problems in fields outside of pure mathematics. This enabled him, in some respects and to some extent, to develop the objective tendencies of immanent logical progress in mathematics, but at the same time it removed him from the real progress of science, acted on emphasizing his differentiation of thinking and his isolatedness more than the Czech medium in which he lived, itself.

Zprávy Komise pro dějiny přírodních, lékařských a technických věd ČSAV, No 11, Praha 1962, pp. 42-45.

42) Bolzano's notes, called Miscellanea mathematica, already mentioned, are best evidence of this extent. Ref. also L. Nový, K rozsahu znalostí zahraniční matematické literatury v českých zemích v první polovině 19. století (On the extent of the knowledge of foreign mathematical literature in the Czech Lands in the first half of the 19th century), in: Zprávy Komise pro dějiny přírodních, lékařských a technických věd ČSAV, No. 9, Praha 1961, pp. 30-31.

43) It would be possible to reconstruct Bolzano's knowledge of literature from Bolzano's notes in the Miscellanea mathematica and from references. Proof of what he actually read, would probably be more difficult.

This deliberation leads us back to the problem of the origins of historical analysis and to the appreciation of Bolzano's mathematical work as a part of the development of science of the 1st half of the 19th century in its long-term tendencies. On the basis of what we have already said about the character and some external associations of Bolzano's work, there appear several very interesting methodological problems in the historical arranging of his mathematical work. Although Bolzano's motivation was certainly determined subjectively and can be explained psychologically, it has an important common feature, i.e. the power and possibilities of immanent logical development of science can be studied on it. In other words, Bolzano's work indicates how progress can be made at a given level of development of mathematics under certain conditions, or what other possibilities this level of development already provided; of course, on the other hand, we can also say to what extent these possibilities were not exploited immediately, but only in a different situation. Examples similiar to Bolzano's, are numerous in the mathematics of the 1st half of the 19th century.[44] Their analyses and generalization would yielded interesting conclusions. However, there is the question as to the extent to which this "degree of free latitude" in mathematical work is typical for the 1st half of the 19th century, or how and possibly why its influence changed in the course of evolution and whether the evolution of mathematics differs substantially just because of the intelectual, logical effort from the evolution and essence of the research deliberations of the natural science disciplines.

Finding concrete relations between the individual parts and ideas of Bolzano's mathematical work and the contemporaneous evolution of mathematics is associated with this methodological problem which deserves a more profound and systematic study. In solving this problem we have to follow several trends. First of all, it is important to pay attention to Bolzano's concrete reactions (selection or continuity) to certain results or authors. This task concerns Bolzano's mathematical work as a whole, and we must admit that a more detailed analysis is still lacking. It has not even touched on such an essential question as Bolzano's relatively small interest and possibly also understanding of Gauss' effort, although we should not doubt that substantial parts of his work were know to Bolzano and that he had also studied some parts of it.[45] Using these and other cases it would be

44) We are of the opinion that this is so with some terms and theories even in geometry and algebra; for example, the way to vectorial understanding of geometry, to noneuclidean geometry or work with quaternions had already been opened up at the turn of the centuries (ref. to: J. Folta, Bernard Bolzano and the Foundation of Geometry, in: Acta historiae rerum naturalium nec non technicarum, Spec. Issue 2, Prague 1966, pp. 75-104; J. Folta, The Foundation of Geometry at the Turn of the 18th and 19th Centuries and Bolzano's Contribution, in: Actes du XI^{e} Congrès International d' Histoire des Sciences, Vol. III., Warszawa 1967); proof of the insolvability of algebraic equations of a degree higher than the fourth led to the explicit study of groups and to other algebraic structures, etc.

45) Bolzano's relation to Gauss would deserve special attention. There is no doubt that Bolzano was acquainted with some of Gauss' works. Indeed, Bolzano's proof of the fundamental lemma of algebra, contained in Rein analytischer Beweis ... (in this edition p. 417 ff.) is founded on the critique of the first three proofs of Gauss of this lemma, published in the years 1799 to 1816, however, it seems that Bolzano did not understand quite well why Gauss chose various approaches. Going by the materials which have been published so far, it is very surprising, how little Bolzano referred to his contemporary. This is also true of the notes in Miscellanea mathematica. It appears that he knew of his results from the Göttingische gelehrte Anzeigen; compare, e.g., with Miscellanea mathematica, Heft 21, p. 1863, where he made a note of Gauss' work Disquisitiones generales circa superficies curvas.

possible to prove and outline Bolzano's method of working with literature and also the early crystallization of his ideas of the content pattern of the interpretation of mathematics, which was then more strongly reflected in the contents and manner of presentation of the intended Grössenlehre.

The other side of the said problem is the question, as to which of Bolzano's problems could be developed further by his contemporaries and to what extent they were actually influenced by Bolzano's ideas. It is evident, moreover, that this question must be considered quite differently with regard to the papers printed in Bolzano's time, and as regards his preserved mathematical manuscripts or orally communicated opinions.[46] This problem is associated with the question of including Bolzano's results in compatible trends of contemporaneous mathematics. Existing historical literature has elaborated on these problems relatively widely, although it was unable to find satisfactory solutions to some

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He also refers to the renowned paper Theoria residuorum biquadraticorum, Commentatio secunda (1832), in Grössenlehre (II A 7, p. 144) according to Gauss' own notification in the Göttingensche gelehrte Anzeigen, in which he explains and justifies the "expansion of the field" of the theory of numbers to the realm of complex numbers, including their geometrical representation. Bolzano did not in fact study Gauss' theory of numbers which had such a profound influence on the development in the 19th century as a whole.

46) The study of how Bolzano's mathematical manuscripts and his results contained in them were known in his and the subsequent period, is still at the very beginning. However, we cannot exclude the possibility that they were known to a certain group of researchers. It is undoubtedly known the Bolzano gave his manuscripts to read, e.g., to Slivka, Zimmermann and others. To what extent it is probable that they were passed on even outside the group of Bolzano's Prague friends is difficult to say, although it cannot be excluded. This was hinted already, e.g., by V. Jarník (Bernard Bolzano and the Foundations of Mathematical Analysis, cf. quot. 9, p. 35).

of the fundamental ones; therefore, there exist different points of view of some of the important partial problems.[47] It is possible that more detailed heuristics of other materials could contribute to solving them. The problem of how well-known the printed treatise of Bolzano from the first period of his mathematical activity were in some foreign centres of mathematical research is similar. It is possible to trace, e.g., the Abhandlungen der königlichen böhmischen Gesellschaft der Wissenschaften of Prague in foreign libraries,[48] rarely one can also find evidence that one or another important mathematician had handled these volumes,[49]

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47) This involves, for example, the question whether Bolzano influenced Cauchy or not. I. Grattan-Guinness has recently devoted attention to this question in his papers: Četl Cauchy Bolzana před napsáním Cours d'analyse (Did Cauchy read Bolzano before he wrote the Cours d'analyse), in: Pokroky matematiky, fyziky a astronomie 15 (1970) No. 3-4, pp. 133-137; Bolzano - Cauchy and the new analysis of the early nineteenth century, in: Archive for history of exact sciences 6 (1970), No. 5, pp. 372-400. He voiced his opinions on the development of mathematical analysis (including an extensive discussion of the Bolzano - Cauchy problem) into an extensive book, The development of the foundations of mathematical analysis from Euler to Riemann, Cambridge MIT 1970. His ideas drew a considerable but not consonant response. Refer, e.g., to H. Freudenthal, Did Cauchy plagiarize Bolzano?, in: Archive for the History of Exact Sciences 7 (1971), pp. 375-392, or H. Sinaceur, Cauchy et Bolzano, in: Revue d'Histoire des Sciences et leurs Applications 26 (1973) No. 2, pp. 97-112.

48) These Abhandlungen were frequently exchanged with other important scientific societies of the whole contemporary scientific world. Whether they were read is another matter, much more difficult to resolve.

49) The discussion of these possibilities is already contained in the papers referred to in quot. 47. For example, it is interesting that N.I. Lobachevskij, in apparently remote Kazan, had handled a volume with Bolzano's Rein analytischer Beweis ... Refer to J. Folta, N.I. Lobačevskij a B. Bolzano (N.I. Lobachevskij and B. Bolzano), in: Zprávy Komise pro dějiny přírodních, lékařských a technických věd ČSAV, No. 8, Praha 1961, pp. 39-40.

however, more direct proof, provided it exists, is not sufficiently known yet. It will be the more difficult to determine how Betrachtungen über einige Gegenstände der Elementargeometrie and Beyträge zu einer begründeteren Darstellung der Mathematik, i.e. the first printed papers of Bolzano, were broadcastened.

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We have tried, at least briefly, to outline the problems and some of the stimuli concerning the historical research of Bolzano's mathematical work. Bolzano's mathematical papers included in this volume, which were printed in the first two decades of the 19th century, will, according to our opinion, attract the interest of historians and mathematicians even in future. In order to faciliate their study, we have published them in their original form. We should thus like to contribute the study of the work of this important thinker, intrinsically associated with the life and complicated problems of the evolution of the Czech society in the 1st half of the 19th century, in the year of Bolzano's anniversary.

Prague, March 25th, 1981
Translated by Jaroslav Tauer

Luboš Nový
Jaroslav Folta

Betrachtungen

über

einige Gegenstände

der

Elementargeometrie

von

Bernard Bolzano.

Τας ἐπιδοσεις ὁρωμεν γιγνομενας, και των τεχνων, και των ἀλλων ἁπαντων, ὐ δια τυς ἐμμενοντας τοις καθεςωσιν, ἀλλα δια τυς ἐπανορθυντας, και τολμωντας ἀει τι κινειν των μη καλως ἐχοντων. Isocr. Evag.

Prag, 1804.
in Commission bey Karl Barth.

Dem

Hochwürdigsten, Hochgelehrten und Wohlgebornen

Herrn, Herrn

Stanislaus Wydra,

Director und Professor der Mathematik, emeritirten Rector Magnificus, Domherrn bey Aller Heiligen etc. etc.

zum Beweise

einer unbegränzten Hochachtung und Dankbarkeit

gewidmet

von seinem ehemaligen Schüler

dem Verfasser.

Einleitende Vorrede.

Es ist nicht unbekannt, daß die Mathematik nebst dem ausgebreiteten Nutzen, den ihre Anwendung auf das praktische Leben gewährt, auch noch einen zweyten kaum geringern, obgleich nicht so in die Sinne fallenden Nutzen durch Übung und Schärfung des Verstandes, durch die wohlthätige Beförderung einer gründlichen Denkart liefern könne; einen Nutzen, welchen der Staat vornehmlich beabsichtigt, wenn er das Studium

die:

dieſer Wiſſenſchaft von jedem Akademiker verlangt. Wie ich nun den kühnen Wunſch nicht unterdrücken konnte, zu dem ſteten Fortſchreiten dieſer ſo vortrefflichen Wiſſenſchaft auch etwas beyzutragen: ſo habe ich — nach meinen ſubjectiven Neigungen — bisher größern Theiles nur die Vervollkommnung der ſpeculativen Mathematik, d. i. der Mathematik, in wiefern ſie den zweyterwähnten Nutzen leiſten ſoll, mir in meinen Nebenſtunden zum Gegenſtande der Betrachtung vorgeſetzt.

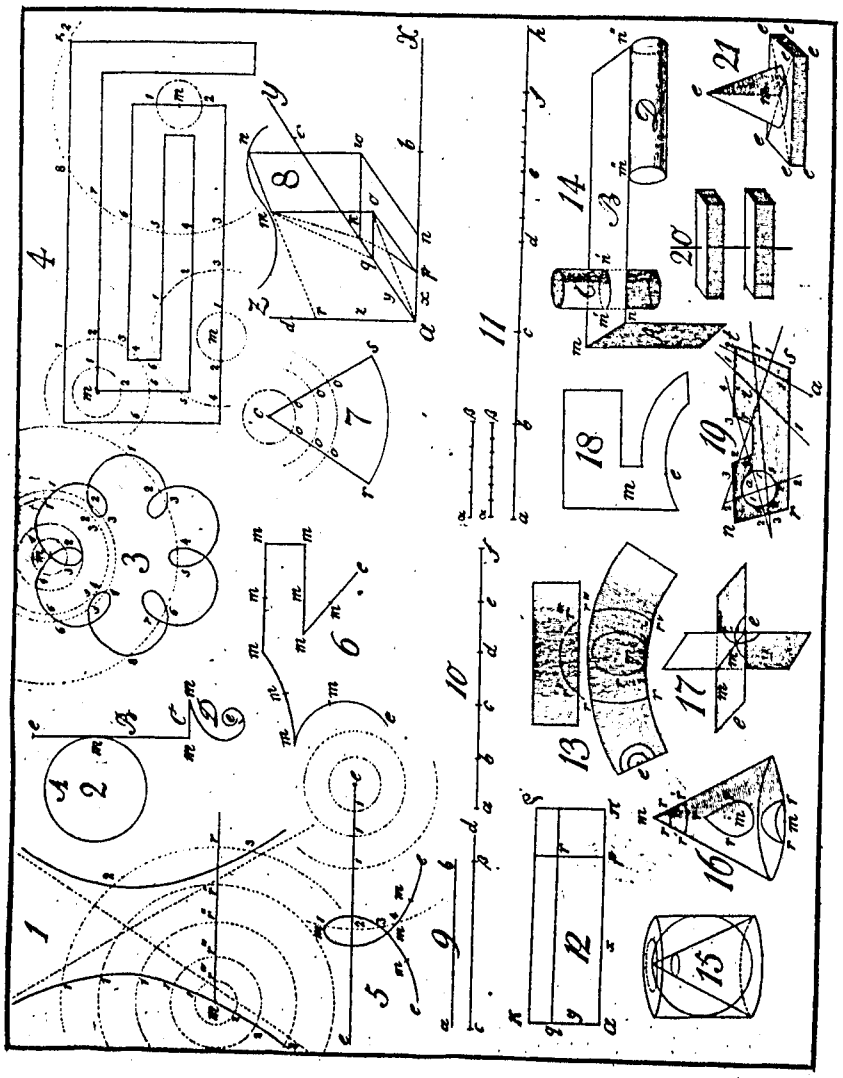
Es iſt nöthig, hier ein Paar der Regeln zu erwähnen, die mir bey dieſem Geſchäfte unter andern nach meiner Meinung oblagen.

Erſtlich ſtellte ich mir die Regel auf, daß ich mich durch keine Evidenz eines Satzes von der Verbindlichkeit los zähle, noch einen Beweis für denſelben aufzuſuchen, — ſo lange, bis ich deutlich einſähe, daß und warum ſich durchaus kein Beweis fernerhin for-

fordern laſſe. Wenn es wahr iſt, daß überall deutliche, richtige, in der vollkommenſten Ordnung verbundne Vorſtellungen leichter zu faſſen ſind, als hie und da noch verworrne und unrichtige: ſo muß man das Beſtreben alle Wahrheiten der Mathematik bis auf ihre letzten Gründe zu entwickeln, und dadurch allen Begriffen dieſer Wiſſenſchaft die möglichſte Deutlichkeit, Berichtigung und Ordnung zu verſchaffen, für ein Beſtreben anſehn, das nebſt der Gründlichkeit auch noch die Leichtigkeit des Unterrichts befördern wird. Und wenn es ferner wahr iſt, daß aus den erſten Vorſtellungen, wenn ſie deutlich und richtig aufgefaßt ſind, auch viel mehreres geſchloſſen werden kann, als wenn ſie noch verworren da liegen: ſo muß dieſem Beſtreben auch zum dritten ein möglicher Nutzen zur Erweiterung der Wiſſenſchaft zugeſtanden werden. Davon gibt die ganze Mathematik die klärſten Beyſpiele. Was konnte einſt überflüſſiger geſchienen haben, als wenn

Tha-

abweichende Art; doch so, daß jener Vorwurf einer Einmengung zufälliger Nebenbegriffe, den wir der Methode der Grenzen (Vorr. S. XVII) gemacht, auch hier noch Statt findet. Oder wer sollte nicht erkennen, daß jene Betrachtung einer zwischen der kleinsten und der größten liegenden Ordinate, die gerade so groß ist, daß ein aus ihr und dem Abscissenzuwachse gebildetes Rechteck dem Zuwachse des krummlinig begrenzten Flächenraumes gleicht, eine sehr fremdartige Betrachtung sey, wenn man blos diesen letzteren berechnen will? — Ein anderer Fehler ist es, daß der Herr Verfasser sich erlaubt, einer Gleichung, deren Gültigkeit nur für alle solche Fälle erwiesen ist, wo die veränderliche Größe einen nicht der Null gleichen Werth hat, gerade auf den Fall, wo dieser Werth
582 gleich Null ist, anzuwenden. Doch dieser Fehler wäre auf die bekannte Art, die im binomischen Lehrsatze §. 28. befolgt ist, leicht zu verbessern.



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