Walks with optimal reward on metric spaces

Ehrhard Behrends

Abstract. Let (M,d) be a complete metric space and suppose that there are given finitely many contractions $\Gamma_\rho : M \to M$ and Lipschitz maps $\varphi_\rho : M \to \mathbb{R}$ ($\rho = 1, \ldots, r$).

We consider "walks" of length m with a given starting point x_0 in M. They are defined as follows: One chooses a sequence $(\rho_\mu)_{\mu=1,\ldots,m}$ of length m in $\{1, \ldots, r\}$, and this choice induces the "walk"

$$x_0,\ x_1 := \Gamma_{\rho_1}(x_0),\ x_2 := \Gamma_{\rho_2}(x_1),\ \ldots,\ x_m := \Gamma_{\rho_m}(x_{m-1}).$$

Associated with $x_1, \ldots, x_m$ is the "reward"

$$\varphi_{\rho_1}(x_0) + \varphi_{\rho_2}(x_1) + \cdots + \varphi_{\rho_m}(x_{m-1}).$$

We denote by $R^{\max}_{x_0}(m)$ the maximal possible reward.

The aim of this note is to investigate the behaviour of the sequence $\bigl(R^{\max}_{x_0}(m)\bigr)$ for large m. It will be shown that the growth is nearly linear: there is a constant γ (which does not depend on x_0) such that $R^{\max}_{x_0}(m)/m$ tends to γ. However, an explicit calculation of γ might be hard. The complexity depends on the fractal dimension of the smallest nonempty compact subset of M which is invariant with respect to all Γ_ρ.

In the case of finite M one can say much more. Then – after a suitable rescaling – the sequence $\bigl(R^{\max}_{x_0}(m)\bigr)$ is periodic where the length of the period can be described in terms of the length of certain cycles of a graph associated with M.

The motivation to study this problem came from a variant of *Parrondo's paradox* from probability theory: What is the optimal choice of games if a great number of players is involved?

keywords: stochastic game, weighted graph, fractal, Parrondo's paradox.

1. Introduction

Parrondo's paradox states that there are losing games which, when combined stochastically in a suitable way, give rise to winning games. For a more precise formulation we use the notation introduced in [3]. A *Parrondo game* consists of a collection $\mathbf{P}_1, \ldots, \mathbf{P}_r$ of stochastic $(s \times s)$-matrices and "reward vectors" $\mathbf{x}_1, \ldots, \mathbf{x}_r \in \mathbb{R}^s$. (We will write $\mathbf{P}_\rho$ as $(p_{ij}^{(\rho)})_{i,j=0,\ldots,s-1}$ and $\mathbf{x}_\rho$ as $(x_i^{(\rho)})_{i=0,\ldots,s-1}$.) One starts the game at $0 \in S := \{0, \ldots, s-1\}$, then a ρ is chosen. One obtains immediately the reward $x_0^{(\rho)}$, and a random step on S is performed according to the probabilities in the first row of $\mathbf{P}_\rho$. Suppose that the resulting state is $i \in S$. Then again a matrix is chosen, say $\mathbf{P}_{\rho'}$. One gets $x_i^{(\rho')}$ and moves according to the i'th row of $\mathbf{P}_{\rho'}$. And so on.

In the case $r = 1$ there is no real choice. One observes that the gain in the μ'th round is the first component of $\mathbf{P}_1^\mu \mathbf{x}_1$, and also that – under the assumption that $\mathbf{P}_1$ is ergodic and μ is not too small – the matrix $\mathbf{P}_1^\mu$ has nearly identical rows each of which approximates the equilibrium $\pi_{\mathbf{P}_1}$ of $\mathbf{P}_1$ (see, e.g., chapter 7 in [3]). Consequently the gain in the μ'th round can be approximated better and better by the scalar product $\langle \pi_{\mathbf{P}_1}, \mathbf{x}_1 \rangle$ of $\pi_{\mathbf{P}_1}$ with $\mathbf{x}_1$ if μ is large, and thus the game should be called *fair* if this scalar product vanishes. Parrondo has observed that there are $(\mathbf{P}_\rho, \mathbf{x}_\rho)$, $\rho = 1, \ldots, r$, such that each individual $(\mathbf{P}_\rho, \mathbf{x}_\rho)$ is fair but it is possible to choose $\rho_1, \ldots, \rho_m$ such that the expected total reward after m rounds tends to infinity with $m \to \infty$.

In [6] Dinis and Parrondo investigate a situation where a huge number N of people play such a Parrondo game: a $\beta \in \,]0, 1]$ is given, and in the μ'th round βN players – which are chosen at random – play their game with $(\mathbf{P}_{\rho_\mu}, \mathbf{x}_{\rho_\mu})$. What is the best choice of $\rho_1, \ldots, \rho_m$?

The first observation is that one may assume that $\beta = 1$: In the case $\beta < 1$ one only has to replace each $\mathbf{P}_\rho$ by $(1-\beta)I + \beta \mathbf{P}_\rho$ and each $\mathbf{x}_\rho$ by $\beta \mathbf{x}_\rho$ (here and in the sequel "I" stands for the identity matrix). Also we note that for the calculation of the collective gain it is only necessary to know the proportions of the players being in state $0, 1, \ldots, s-1$. Suppose that in the μ'th round these are $v_0^{(\mu)}, \ldots, v_{s-1}^{(\mu)}$. Then the collective gain in this round is

$$N\big(v_0^{(\mu)} x_0^{(\rho_\mu)} + \cdots + v_{s-1}^{(\mu)} x_{s-1}^{(\rho_\mu)}\big),$$

i.e., N times the scalar product of $\mathbf{v}^{(\mu)} := (v_0^{(\mu)}, \ldots, v_{s-1}^{(\mu)})$ with $\mathbf{x}_\rho$. Also, after this round, the new proportions in the states $0, \ldots, s-1$ will be the components of the vector $\mathbf{v}^{(\mu)} \mathbf{P}_\rho$.

In order to avoid that the gain grows over all bounds with $N \to \infty$ it will be appropriate to rescale the $\mathbf{x}_\rho$: in the case of N players we replace these gain vectors by $\mathbf{x}_\rho/N$. Then we arrive at the following problem: Let Δ_s be the collection of all probability vectors in $\mathbb{R}^s$. For a $\mathbf{v} \in \Delta_s$ and $\rho = 1, \ldots, r$ we define

$$\Gamma_\rho(\mathbf{v}) := \mathbf{v}\mathbf{P}_\rho \in \Delta,\ \varphi_\rho(\mathbf{v}) := \langle \mathbf{v}, \mathbf{x}_\rho \rangle \in \mathbb{R}.$$

One wants to know which choice of $\rho_1, \ldots, \rho_m$ gives rise to the maximal collective gain if m – the number of rounds – and the starting distribution $\mathbf{v}_0$ are prescribed. It remains to note that it is generally assumed that the $\mathbf{P}_\rho$ are not only *ergodic* but that one has some quantitative information about the ergodic behaviour. One assumes that there is a number $L < 1$ such that the l^1-distance between two arbitrary rows of any $\mathbf{P}_\rho$ is bounded by $2L$:

$$\sum_j \left| p_{ij}^{(\rho)} - p_{i'j}^{(\rho')} \right| \leq 2L$$

for $i, i' = 0, \ldots, s-1$ and $\rho, \rho' = 1, \ldots, r$. As a consequence of this condition the mappings Γ_ρ are contractions with Lipschitz constant L on Γ_s (see, e.g., lemma 10.6 in [3]). Since, as a consequence of the Cauchy-Schwarz inequality, the φ_ρ are Lipschitz maps, we therefore are precisely in the situation described in the abstract. M is the compact space Δ_s, provided with the l^1-distance.

The paper will be organized as follows. We start in *section 2* with some supplements concerning the precise description of our problem. Then we note that it can be thought of as the search for optimal walks in a certain directed weighted graph. The vertices of this graph are the points of M, an essential role will play the smallest closed nonvoid subset F of M which is invariant with respect to all Γ_ρ. The set F has in many cases a fractal structure.

In *section 3* we restrict our attention to the case of *finite* M. We describe the behaviour of the sequence $\left(R_{x_0}^{\max}(m)\right)$ completely in the slightly more general setting of finite graphs. With the help of elementary number theory one can prove that this sequence is "periodic", the period can be rather large.

The methods developed in section 3 will be used in *section 4* to treat the general case. As for finite M there is a constant γ which is something like the "value of the game": if one plays in an optimal way, then the gain per round is essentially γ. The compactness of M plays an essential role, it enables us to approximate the infinite problem by a finite situation.

The question remains how to determine γ numerically. This is surprisingly complicated, a result by which approximations can be obtained is given in *section 5*. It will be shown that the complexity depends on the fractal dimension of F.

2. Preliminaries

The meaning of (M,d), the Γ_ρ, the φ_ρ $(\rho = 1,\ldots,r)$ and $R^{\max}_{x_0}(m)$ will be as introduced in the abstract. The Γ_ρ are contractions, we will denote by L the maximum of their contraction constants. Thus $0 \le L < 1$, and $d\big(\Gamma_\rho(x),\Gamma_\rho(y)\big) \le L\, d(x,y))$ for arbitrary x,y and ρ. Also the φ_ρ are Lipschitz maps. Let L' be a number such that always $|\varphi_\rho(x) - \varphi_\rho(y)| \le L'\, d(x,y)$ holds.

As in the abstract the Γ_ρ will be thought of as "moves" of a game, and the φ_ρ are "reward functions". We are interested in rewards associated with walks starting at x_0 which are induced by the choices $\rho_1,\ldots,\rho_m \in \{1,\ldots,r\}$. We will call this number $R(x_0;\rho_1,\ldots,\rho_m)$:

$$R(x_0;\rho_1,\ldots,\rho_m) := \varphi_{\rho_1}(x_0) + \varphi_{\rho_2}(x_1) + \cdots + \varphi_{\rho_m}(x_{m-1}),$$

where $x_{k+1} := \Gamma_{\rho_{k+1}}(x_k)$ for $k = 1,\ldots,m-1$. With this notation $R^{\max}_{x_0}(m)$ is the maximum of the r^m numbers $R(x_0;\rho_1,\ldots,\rho_m)$.

We want to investigate how the $R^{\max}_{x_0}(m)$ behave for large m and how one can determine the $\rho_1,\ldots,\rho_m$ which give rise to the best choice.

The set F of fixed points

Recall that, by Banach's fixed point theorem, contractions on complete metric spaces have a unique fixed point and that these fixed points are stable. For $\rho_1,\ldots,\rho_l \in \{1,\ldots,r\}$ the map $\Gamma_{\rho_1}\circ\cdots\circ\Gamma_{\rho_l}$ is a contraction (with contraction constant L^l) on M, and thus there exists a unique $\pi_{\rho_1\ldots\rho_l}$ in M such that

$$\Gamma_{\rho_1}\circ\cdots\circ\Gamma_{\rho_l}(\pi_{\rho_1\ldots\rho_l}) = \pi_{\rho_1\ldots\rho_l}.$$

The results of the following lemma are "folklore". They are contained here for the sake of completeness.

Lemma 2.1.

(i) *Let $x \in M$ be such that $d\big(\Gamma_{\rho_1}\circ\cdots\circ\Gamma_{\rho_l}x, x\big) \le \eta$ for some number $\eta \ge 0$. Then $d(x,\pi_{\rho_1\ldots\rho_l}) \le \eta/(1-L^l)$.*

(ii) *If $(\rho_l)_l$ is a sequence in $\{1,\ldots,r\}$, then $(\pi_{\rho_1\ldots\rho_l})_l$ converges in M. We will denote by $\pi_{\rho_1\rho_2\ldots}$ the limit of this sequence.*

(iii) $\Gamma_\rho(\pi_{\rho_1\rho_2\ldots}) = \pi_{\rho\rho_1\rho_2\ldots}$ *for arbitrary* $\rho, \rho_1, \rho_2, \ldots$

(iv) $\Gamma_{\rho_l}(\pi_{\rho_1\ldots\rho_l}) = \pi_{\rho_l\rho_1\ldots\rho_{l-1}}$.

(v) *Consider the collection* $\{1,\ldots,r\}^{\mathbb{N}}$ *of all sequences in* $\{1,\ldots,r\}$, *we will provide this set with the product topology. We claim that the map*

$$\Phi : \{1,\ldots,r\}^{\mathbb{N}} \to M,\ (\rho_l)_l \mapsto \pi_{\rho_1\rho_2\ldots}$$

is continuous. Thus, since $\{1,\ldots,r\}^{\mathbb{N}}$ *is compact, it follows that the image* F *of* Φ *is a compact subset of* M.

Remark: One has to distinguish carefully between the $\pi_{\rho_1\ldots\rho_l}$ (finitely many indices) and the $\pi_{\rho_1\ldots}$ (infinitely many indices). We note that $\pi_{\rho_1\ldots\rho_l}$ coincides with $\pi_{\rho_1\ldots\rho_l\rho_1\ldots\rho_l\rho_1\ldots\rho_l\ldots}$ (the $\rho_1\ldots\rho_l$ are repeated infinitely often).

Proof: (i) This is a special case of a general result for contractions. Suppose that T is a contraction with Lipschitz constant $\lambda < 1$ and fixed point x_0 and that $d(Tx, x) \leq \eta$. Then

$$\begin{aligned} d(T^k x, x) &\leq d(T^k x, T^{k-1}x) + d(T^{k-1}x, T^{k-2}x) + \cdots + d(Tx, x) \\ &\leq (\lambda^{k-1} + \cdots + \lambda + 1) d(Tx, x) \\ &\leq (1 + \lambda + \lambda^2 + \cdots)\eta \\ &= \eta/(1-\lambda). \end{aligned}$$

The $T^k x$ converge to x_0 so that, by continuity, $d(x_0, x) \leq \eta/(1-\lambda)$.
This has to applied here with $T = \Gamma_{\rho_1} \circ \cdots \circ \Gamma_{\rho_l}$.

(ii) Let $(\rho)_l$ be a sequence in $\{1,\ldots,r\}$, we will show that $(\pi_{\rho_1\ldots\rho_l})_l$ is a Cauchy sequence. To this end, let $\varepsilon > 0$ be given. We choose $l_0 \in \mathbb{N}$ such that L^{l_0} times the diameter of M is smaller than ε. This implies that the diameter of the range R of $\Gamma_{\rho_1} \circ \cdots \circ \Gamma_{\rho_{l_0}}$ is bounded by ε. It remains to note that $\pi_{\rho_1\ldots\rho_l}$ and $\pi_{\rho_1\ldots\rho_{l'}}$ lie in R for $l, l' \geq l_0$ so that

$$d(\pi_{\rho_1\ldots\rho_l}, \pi_{\rho_1\ldots\rho_{l'}}) \leq \varepsilon.$$

The completeness of M implies that $(\pi_{\rho_1\ldots\rho_l})_l$ converges.

(iii) Let $\varepsilon > 0$ be given. We choose l_0 such that the diameter of the range R of $\Gamma_{\rho_1} \circ \cdots \circ \Gamma_{\rho_{l_0}}$ is bounded by ε. Then $\pi_{\rho_1\rho_2\ldots}$ lies in R. Further, the diameter of $\Gamma_\rho(R)$ is at most $L\varepsilon$, and both $\Gamma_\rho(\pi_{\rho_1\rho_2\ldots})$ and $\pi_{\rho\rho_1\rho_2\ldots}$ are contained in this set. It follows that

$$d\big(\Gamma_\rho(\pi_{\rho_1\rho_2\ldots}), \pi_{\rho\rho_1\rho_2\ldots}\big) \leq L\varepsilon,$$

and the result follows since ε was arbitrary.

(iv) By definition we know that

$$\Gamma_{\rho_1} \circ \cdots \circ \Gamma_{\rho_l} \pi_{\rho_1 \ldots \rho_l} = \pi_{\rho_1 \ldots \rho_l}.$$

If we apply Γ_{ρ_l} to this identity it follows that $\Gamma_{\rho_l}(\pi_{\rho_1 \ldots \rho_l})$ is a fixed point of $\Gamma_{\rho_l \rho_1 \cdots \rho_{l-1}}$. Since this fixed point is uniquely determined we may conclude that $\Gamma_{\rho_l}(\pi_{\rho_1 \ldots \rho_l}) = \pi_{\rho_l \rho_1 \ldots \rho_{l-1}}$.

(v) Let $\rho_1, \rho_2, \ldots$ and $\varepsilon > 0$ be given. If l_0 is such that the diameter of the range of $\Gamma_{\rho_1} \circ \cdots \circ \Gamma_{\rho_{l_0}}$ is at most ε, then $d(\pi_{\rho_1 \rho_2 \ldots}, \pi_{\rho'_1 \rho'_2 \ldots}) \leq \varepsilon$ provided that the first l_0 terms of $(\rho_1, \rho_2, \ldots)$ and $(\rho'_1, \rho'_2, \ldots)$ coincide. Since

$$\{(\rho'_1, \rho'_2, \ldots) \mid \rho'_i = \rho_i \text{ for } i = 1, \ldots, l_0\}$$

is a neighbourhood of $\rho_1, \rho_2, \ldots$ with respect to the product topology this proves the continuity of Φ. □

Denote by F the range of the mapping Φ from part (v) of the preceding lemma. This set will play an important role in the sequel. The examples which will be discussed in the next subsection indicate that F often has a *fractal structure*[1].

Lemma 2.2.

(i) *F is the smallest compact nonempty subset of M which is invariant with respect to all Γ_ρ.*

(ii) *Let M_l be the union of all $\Gamma_{\rho_1} \circ \cdots \circ \Gamma_{\rho_l}(M)$, where $\rho_1, \ldots, \rho_l$ run through $\{1, \ldots, r\}$. Then $F = \bigcap_l M_l$.*

(iii) *For every $x \in M$ and arbitrary $\rho_1, \rho_2, \ldots$ the $\Gamma_{\rho_l} \circ \cdots \circ \Gamma_{\rho_1}(x)$ tend with $l \to \infty$ to F: for every $\varepsilon > 0$ there is an l_0 such that*

$$d\big(\Gamma_{\rho_l} \circ \cdots \circ \Gamma_{\rho_1}(x), F\big) \leq \varepsilon$$

for $l \geq l_0$.

Remark: Given a sequence $\rho_1, \rho_2, \ldots$ we will have to deal with products of the form $\Gamma_{\rho_1} \circ \cdots \circ \Gamma_{\rho_l}$ and also of the form $\Gamma_{\rho_l} \circ \cdots \circ \Gamma_{\rho_1}$ for increassing l. The first variant has been used in lemma 2.1, there the sequence $(\pi_{\rho_1 \cdots \rho_l})_{l=1,2,\ldots}$ was of importance, where $\pi_{\rho_1 \cdots \rho_l}$ is the equilibrium of $\Gamma_{\rho_1} \circ \cdots \circ \Gamma_{\rho_l}$.

If, however, one is interested in the orbit of the starting point x_0 one has to investigate the $\big(\Gamma_{\rho_l} \circ \cdots \circ \Gamma_{\rho_1}\big)(x_0)$.

[1] We note that already in [1] fractal phenomena in connection with Parrondo's games have been described. In particular this paper contains an analysis of the one-dimensional situation. There is only a slight overlap with the present investigations.

Proof: (i) By the preceding lemma F is compact and invariant with respect to all Γ_ρ. Conversely, if $F' \subset M$ is a nonempty closed subset which is left invariant by all Γ_ρ it has contain all $\pi_{\rho_1 \dots \rho_l}$: one only has to note that

$$\pi_{\rho_1 \dots \rho_l} = \lim_k \bigl(\Gamma_{\rho_1} \circ \cdots \circ \Gamma_{\rho_l}\bigr)^k(x)$$

for arbitrary x. But then also the $\pi_{\rho_1 \rho_2 \dots}$ which lie in the closure of the set of $\pi_{\rho_1 \dots \rho_l}$ are in F'. This proves that $F \subset F'$.

(ii) The set $F' := \bigcap_l M_l$ is closed and invariant with respect to the Γ_ρ so that $F \subset F'$. On the other hand, if the diameter of $\Gamma_{\rho_1} \circ \cdots \circ \Gamma_{\rho_l}(M)$ is bounded by ε, then all elements of this set are ε-close to $\pi_{\rho_1 \cdots \rho_l} \in F$. It follows that – for arbitrary ε – all $x \in F'$ are ε-close to some point in F, and this yields $F' \subset F$.

(iii) This assertion follows immediately from the preceding proof. □

Examples/Remarks

1. First we consider the case $r = 1$. There is only one contraction $\Gamma_1 = \Gamma$ and only one $\varphi_1 = \varphi$ and thus

$$R_{x_0}^{\max}(m) = \varphi(x_0) + \varphi\bigl(\Gamma(x_0)\bigr) + \varphi\bigl(\Gamma^2(x_0)\bigr) + \cdots + \varphi\bigl(\Gamma^{m-1}(x_0)\bigr).$$

The $\Gamma^k(x_0)$ tend geometrically fast to the fixed point x' of Γ. Therefore, since φ is Lipschitz, the summands tend fast to $\varphi(x')$: one has

$$\Bigl|\varphi(x') - \varphi\bigl(\Gamma^k(x_0)\bigr)\Bigr| \le CL'L^k,$$

where C is a constant. It follows that $\bigl|R_{x_0}^{\max}(m) - m\varphi(x')\bigr| \le mCL'/(1-L)$, and in particular one has

$$\lim_m \frac{R_{x_0}^{\max}(m)}{m} = \varphi(x').$$

Note that F in this case is the singleton $\{x'\}$.

2. Suppose that $L = 0$, i.e., all Γ_ρ are constant maps. Let x'_ρ be the fixed point of Γ_ρ. It is clear that $F = \{x'_1, \dots, x'_r\}$ and that

$$R(x_0; \rho_1, \dots, \rho_m) := \varphi_{\rho_1}(x_0) + \varphi_{\rho_2}(x'_{\rho_1}) + \cdots + \varphi_{\rho_m}(x'_{\rho_{m-1}})$$

in this case. It is remarkable that already in this special situation it is not obvious how to choose $\rho_1, \dots, \rho_m$ such that the reward is maximal.

The easiest way is to solve this problem is by a *backwards analysis*. Denote, for $\rho = 1, \ldots, r$ and $k \geq 0$ by G_ρ^k the maximal possible gain when one starts at x'_ρ and k rounds are to be played. Then $G_\rho^0 = 0$, and

$$G_\rho^{k+1} = \max_\nu \bigl(\varphi_\nu(x'_\rho) + G_\nu^k\bigr).$$

It remains to note that $R_{x_0}^{\max} = \max_\rho \bigl(\varphi_\rho(x_0) + G_\rho^{m-1}\bigr)$.

Even in that special situation one can observe a phenomenon which could be thought of as *a variant of Parrondo's paradox* (which is, as has to be admitted, far from being spectacular). Call a pair $(\Gamma_\rho, \varphi_\rho)$ *fair* if φ_ρ is zero at the fixed point of Γ_ρ. This notation is justified by the observations in connection with the preceding example 1. It is now easy to find Γ's and φ's such that (Γ_1, φ_1) and (Γ_2, φ_2) are fair but $R_{x_0}^{\max}(m)/m$ tends to a positive (or negative) number. One simply chooses the Γ_ρ to be constant with (different) fixed points x_1, x_2 and defines φ_1 (resp. φ_2) to be zero at x_1 (resp. at x_2)

3. Let $\alpha, \beta \in \,]\,0, 1\,[$ be fixed. We consider $M = [\,0, 1\,]$ with the usual metric and the two contractions $\Gamma_1 : x \mapsto \beta x$, $\Gamma_2 : x \mapsto (1 - \beta) + \beta x$ on M. The maps φ_1, φ_2 are defined by $\varphi_1(x) := x - \alpha$ and $\varphi_2(x) := 0$.

The set F will depend on β. For $\beta \in [\,0.5, 1\,[$ one has $F = [\,0, 1\,]$, but for $\beta \in \,]\,0, 0.5\,[$ the set F is fractal-like. (E.g., for $\beta = 1/3$ one obtains the usual Cantor set.)

Consider the starting point $x_0 = 0$. What is, for a given m, the optimal choice of the $\rho_1, \ldots, \rho_m$? For the first step it is surely better to deal with $\rho_1 = 2$ than with $\rho_1 = 1$ since in the second case the gain is negative and one would stay at 0. Surely it would be better to choose $\rho = 2$ for some rounds: at least so often that $x_k = (\Gamma_2)^k(0) > \alpha$. If then $\rho = 1$ is chosen, one obtains $x_k - \alpha$. It is not clear, however, whether it would not be wiser to stay at $\rho = 2$ for some further steps: the gain would still be be zero, but one would arrive at points where the gain with the choice $\rho = 1$ is much better than $\alpha - x_k$.

4. The present investigations have been motivated by collective Parrondo games. As explained in the introduction they are defined by a family of r stochastic $(s \times s)$-matrices $\mathbf{P}_1, \ldots, \mathbf{P}_r$ and vectors $\mathbf{x}_1, \ldots, \mathbf{x}_r \in \mathbb{R}^s$. The metric space M is the collection of all probability measures on $S = \{0, \ldots, s-1\}$, and the maps Γ_ρ and φ_ρ are defined by $\mathbf{v} \mapsto \mathbf{v}\mathbf{P}_\rho$ and $\mathbf{v} \mapsto \langle \mathbf{v}, \mathbf{x}_\rho \rangle$.

We will restrict ourselves here to the case $s = 3$. Then M consists of the (p_0, p_1, p_2) such that $p_0, p_1, p_3 \geq 0$ and $p_0 + p_1 + p_2 = 1$. This collection will be represented by the points of an equilateral triangle with *barycentric* *coordinates*: $(1, 0, 0)$, $(0, 1, 0)$ and $0, 0, 1$ are mapped to the bottom left, the bottom right and the top corner, respectively, and the the map, which assigns a probability to a point is affine[2].

Depending on the situation the subset F can look rather differently. Here are three examples:

a) First we consider Parrondo's original example. The matrices are

$$\mathbf{P}_1 = \begin{pmatrix} 0 & 0.5 & 0.5 \\ 0.5 & 0 & 0.5 \\ 0.5 & 0.5 & 0 \end{pmatrix}, \ \mathbf{P}_2 = \begin{pmatrix} 0 & 0.1 & 0.9 \\ 0.25 & 0 & 0.75 \\ 0.75 & 0.25 & 0 \end{pmatrix}.$$

In barycentric coordinates the associated fractal looks like this[3]:

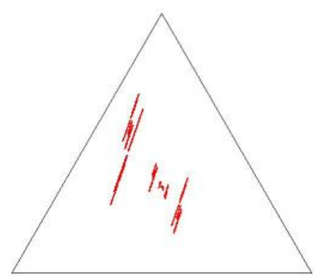

fig. 1: The fractal associated with Parrondo's original example

(We note that a picture of this fractal also can be found in [9]. Also there barycentric coordinates are used.)

b) For the next example the stochastic matrices have been produced by a random generator:

$$\mathbf{P}_1 = \begin{pmatrix} 0,000250 & 0,499957 & 0,499793 \\ 0,499785 & 0,000353 & 0,499862 \\ 0,000576 & 0,999030 & 0,000394 \end{pmatrix}, \ \mathbf{P}_2 = \begin{pmatrix} 0,762578 & 0,004199 & 0,233224 \\ 0,333454 & 0,333209 & 0,333338 \\ 0,227731 & 0,037911 & 0,734358 \end{pmatrix}.$$

Here the fractal F has the following form.

[2]For example, the equidistribution is mapped to the midpoint of the triangle, and probabilites where p_2 is "large" will be found "close" to the top corner.

[3]Note that for the description of F it is not necessary to know the gain vectors.

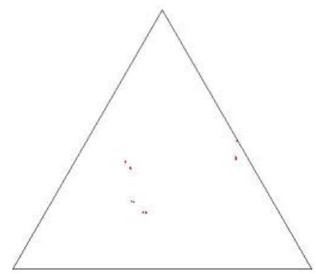

fig. 2: A "sparse" fractal

c) Also in our third example $\mathbf{P}_1$ and $\mathbf{P}_2$ are random stochastic matrices:

$$\mathbf{P}_1 = \begin{pmatrix} 0,000050 & 0,000333 & 0,999616 \\ 0,000252 & 0,499960 & 0,499789 \\ 0,999414 & 0,000574 & 0,000012 \end{pmatrix}, \ \mathbf{P}_2 = \begin{pmatrix} 0,000364 & 0,999632 & 0,000005 \\ 0,000934 & 0,000029 & 0,999037 \\ 0,499681 & 0,500060 & 0,000259 \end{pmatrix}.$$

They generate the following F:

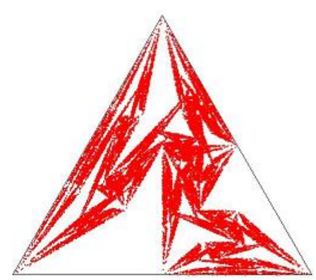

fig. 3: A more complicated fractal

The examples indicate that it is hard to predict from the $\mathbf{P}_\rho$ how F might look. As a vague rule one only can assert that "interesting" F are unlikely to occur when the $\mathbf{P}_\rho$ are strongly mixing, i.e., when the Lipschitz constant of the associated maps Γ_ρ is small. The second example shows that the reverse needs not be true, there the ergodicity constant is close to one, but they give rise to a rather small F.

5. Let $\varepsilon > 0$ and let x_0 be such that there exists a $y_0 \in F$ with $d(x_0, y_0) < \varepsilon$. Then, for any $\rho_1, \ldots, \rho_l$, the distance between the points $\Gamma_{\rho_l} \circ \cdots \circ \Gamma_{\rho_1}(x_0)$ and $\Gamma_{\rho_l} \circ \cdots \circ \Gamma_{\rho_1}(y_0)$ is at most $\varepsilon \cdot L^l$. Therefore

$$\left|\varphi_\rho\big(\Gamma_{\rho_l} \circ \cdots \circ \Gamma_{\rho_1}(x_0)\big) - \varphi_\rho\big(\Gamma_{\rho_l} \circ \cdots \circ \Gamma_{\rho_1}(y_0)\big)\right| \le \varepsilon \cdot L^l \cdot L',$$

and it follows that

$$|R(x_0;\rho_1,\dots,\rho_m)-R(y_0;\rho_1,\dots,\rho_m)|\le\varepsilon\frac{L'}{1-L}.$$

This is true for arbitrary $\rho_1,\dots,\rho_m$ so that

$$\left|R^{\max}_{x_0}(m)-R^{\max}_{y_0}(m)\right|\le\varepsilon\frac{L'}{1-L}.$$

If one combines this observation with lemma 2.2(ii) one may conclude that for every $x_0\in M$ there is a $y_0\in F$ such that

$$\lim_{m\to\infty}\frac{R^{\max}_{x_0}(m)}{m}-\frac{R^{\max}_{y_0}(m)}{m}=0.$$

Thus, if one wants to determine the long-term behaviour of the sequences $(R^{\max}_{x_0}(m))_m$ it will suffices to investigate the $x_0\in F$.

Graphs

In order to visualize our problem of describing $R^{\max}_{x_0}(m)$ it will be helpful to use the language of *graph theory*. Think of M as the collection of vertices of a graph which will in general be infinite. For $x\in M$ there are r directed edges, namely to the points $\Gamma_1(x),\dots,\Gamma_r(x)$. To each of these edges we associate the weight $\varphi_\rho(x)$. In this translation one has to solve the following problem:

> Given $m\in\mathbb{N}$ and $x_0\in M$, find a walk of length m which starts at x_0 such that the total weight is as large as possible.

In the next section we will solve this problem in the case of finite graphs completely, the same ideas will be used later to treat the general case by suitable approximations.

It has been noted above that it often suffices to consider the $x_0\in F$. Since F is invariant with respect to all Γ_ρ this set can be thought of as a subgraph. As a graph F is "nearly connected":

Lemma 2.3. *For all $x_0,y_0\in F$ and every $\varepsilon>0$ there is a walk which starts at x_0 and ends ε-close to y_0: there are $\rho_1,\dots,\rho_d$ such that*

$$d\big(\Gamma_{\rho_d}\circ\cdots\circ\Gamma_{\rho_1}(x_0),y_0\big)\le\varepsilon.$$

It follows that F, considered as a graph, is connected if F is finite.

Proof: Choose $\pi_{\rho_1\ldots\rho_l}$ such that $d(y_0,\pi_{\rho_1\ldots\rho_l})\le\varepsilon/2$ and an l' such that $L^{ll'}$ times the diameter of M is bounded by $\varepsilon/2$. Then the diameter δ of the range of $\big(\Gamma_{\rho_1}\circ\cdots\circ\Gamma_{\rho_l}\big)^{l'}$ satisfies $\delta\le\varepsilon/2$, and this range contains $\pi_{\rho_1\ldots\rho_l}$. Thus

$$d\Big(\big(\Gamma_{\rho_1}\circ\cdots\circ\Gamma_{\rho_l}\big)^{l'}(x_0),y_0\Big)\le\varepsilon.$$

$\square$

3. Optimal paths on finite directed weighted graphs

Let $G=(V,E)$ be a finite directed weighted graph: V is the (finite) set of vertices, $E\subset V\times V$ is the set of edges, and the weight of an $e\in E$ is denoted by w_e. We assume that G is connected, in view of lemma 2.3 this will be no restriction in the present context. Since we will be interested in maximal total gains only it will be no restriction to assume that there is always at most one directed edge connecting two given vertices: if there should be more cancel all but one with maximal weight. Such graphs can be sketched as follows:

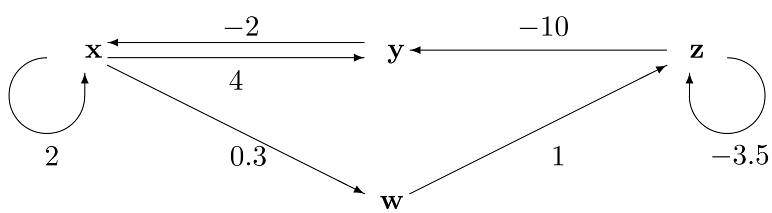

fig. 4

Let x_0 be a fixed vertex, we consider walks of length m which start at x_0. (A *walk W of length n* is a sequence $y_0,y_1,\ldots,y_n$ of vertices such that each two consecutive members of this sequence define a directed edge.) The *gain G_W* of such a walk W is the sum of the weights of the edges which are passed, and we are interested in the *maximal possible gain*. This number will be denoted by $R^{\max}_{x_0}(m)$.[4]

The aim of this section is to describe how the sequence $\big(R^{\max}_{x_0}(m)\big)_{m=1,\ldots}$ behaves for large m, this will prepare the investigations of the general case.

[4] It is justified to use the same notation as in the case of the $M,\Gamma_\rho,\varphi_\rho$ since for finite M we arrive at this graph problem.

We have not found results concerning this problem in the standard text books of graph theory. The reason might be that in the theory of weighted graphs it is more interesting to find walks of minimal (or maximal) total gain which connect two vertices than to consider walks of a given length.

We begin our discussion with some further *notation*. A *cycle (of length n)* C is a walk $y_0, \ldots, y_n$ where $y_0 = y_n$ and where the vertices $y_1, \ldots, y_n$ are pairwise different. If C is such a cycle, the *cycle gain* G_C is defined as the sum over the weights of the edges contained in C, i.e.

$$G_C := w_{\{y_0,y_1\}} + \cdots + w_{\{y_{n-1},y_n\}}.$$

G_C, devided by the length n of C, is the *stepsize gain* associated with C. This number will be called γ_C.

There are only finitely many cycles in G, and therefore the maximum γ over the γ_C exists. γ is the *maximal stepsize gain*, this number will play an important role in the sequel. Since every closed walk can be built up from cycles the number γ is also the maximal stepsize gain for the much larger family of closed walks.

We will write $\mathcal{C}$ for the collection of all cycles and $\mathcal{C}'$ for the subcollection of $\mathcal{C}$ consisting of all C with $\gamma_C = \gamma$.

It can happen that $\mathcal{C} = \mathcal{C}'$, but usually $\mathcal{C}'$ is much smaller than $\mathcal{C}$. In most cases $\mathcal{C}'$ will even contain only one element. (For example, in the graph of the above picture one has $\gamma = 2$ and $\mathcal{C}'$ consists only of the self-loop at $\mathbf{x}$.)

The main result of this section states that the sequence $R^{\max}_{x_0}(\cdot)$ behaves rather regularly:

Proposition 3.1. *There are $l_0, m_0 \in \mathbb{N}$ such that*

$$R^{\max}_{x_0}(m + l_0) = R^{\max}_{x_0}(m) + \gamma \cdot l_0$$

for $m \geq m_0$. In particular $\left(R^{\max}_{x_0}(m)\right)_m$ is "finally periodic" if $\gamma = 0$.

Proof: Let l_0 be the smallest common divisor of the lengths of the cycles in $\mathcal{C}'$. We will show that the assertion holds with this l_0 and sufficiently large m. We may restrict our attention to the case $\gamma = 0$ since the transformation $w_e \mapsto w_e - \gamma$ leads immediately to this situation. This will be assumed from now on.

First we show that $R^{\max}_{x_0}(m)$ is "not too small". Choose any $C_0 \in \mathcal{C}'$ and define – for given "large" m – a walk W in the following way: move directly to C_0 and then keep walking around C_0 until m steps are completed. The gain associated with W is the gain of a walk of at most $N = \operatorname{card} V$ steps (from x_0 to C_0 plus – maybe – some steps in C_0 if the final round is not completed) plus several complete rounds in C_0 which by our assumption have gain zero. (Recall that "card" stands for "cardinality": N is just the number of points in V.) Thus G_W and consequently also $R_{x_0}^{\max}(m)$ is bounded from below by the constant $K_1 := N \min_{e \in E} w_e$.

Next we prove that walks with maximal gain cannot waste too much time in the cycles $C \in \mathcal{C}'' := \mathcal{C} \setminus \mathcal{C}'$.
Define a strictly positive number δ by

$$\delta := - \max_{C \in \mathcal{C}''} \gamma_C;$$

then, by definition, $G_C \leq -n\delta$ for all cycles of length n in $\mathcal{C}''$.

We consider any walk W of length m which starts at x_0. Write W as $x_0 x_1 \cdots x_m$, where x_i, x_{i+1} are adjacent vertices. Suppose that – for certain $k < l$ – one has $x_k = x_l$ such that the $x_k, x_{k+1}, \ldots, x_{l-1}$ are pairwise different. Then G_W is the sum of the gain of

$$W' := x_0 x_1 \cdots x_k x_{l+1} \cdots x_m$$

plus the gain of the cycle-walk $x_k x_{k+1} \cdots x_{l-1}$. The latter number is zero (if the cycle is in $\mathcal{C}'$) or bounded from above by $-\delta(l-k)$ otherwise. If we apply this observation to the "reduced" walk W' and continue until we obtain a walk without cycles we arrive at the following fact:

> Let n_W be the number of steps of the walk which are used for complete cycle-rounds with cycles from $\mathcal{C}''$. Then
>
> $$G_W \leq -n_W \delta + N \max_e w_e.$$
>
> One only has to note that the remaining walk (when there is no further reduction by omitting cycles possible) has at most N steps so that its weight is bounded by $N \max_e w_e$.

So far we have shown that

$$N \min_{e \in E} w_e \leq -n_W \delta + N \max_{e \in E} w_e$$

whenever W is a walk of length m such that $G_W = R_{x_0}^{\max}(m)$, and this means that for such walks n_W is bounded by the constant

$$K := (N \max_{e \in E} w_e - N \min_{e \in E} w_e)/\delta$$

which does not depend on m.

Let $\mathcal{W}$ be the collection of walks which contain no cycles in $\mathcal{C}'$ and for which the number of steps which are used for complete cycle-rounds in cycles from $\mathcal{C}''$ is bounded by K. We know already that each W with $G_W = R_{x_0}^{\max}(m)$ is built up from a $\tilde{W} \in \mathcal{W}$ to which there are "attached" various cycles in $\mathcal{C}'$.

Let us have a closer look at the $\tilde{W} \in \mathcal{W}$. For every such $\tilde{W}$ its length $l_{\tilde{W}}$ is bounded by $N + K$, in particular the set $\mathcal{W}$ must be *finite*. If a $\tilde{W} \in \mathcal{W}$ touches cycles $C \in \mathcal{C}'$ of length $n_1, \ldots, n_k$ then one can use $\tilde{W}$ to construct walks of length $l_{\tilde{W}}$ (the length of $\tilde{W}$) plus a multiple of n_1 plus a multiple of $n_2 \ldots$ plus a multiple of n_k: simply add to W the appropriate number of rounds in the $\mathcal{C}'$-cycles. These new walks all will have gain $G_{\tilde{W}}$.

This has the following consequence: if $\mathbb{N}_{\tilde{W}}$ denotes the collection of all integers m such that there is a walk W (starting at x_0) of length m with $G_W = R_{x_0}^{\max}(m)$ for which the procedure described above (cancel all cycle walks in $\mathcal{C}'$) leads to $\tilde{W}$, then

$$\mathbb{N}_{\tilde{W}} + n_1\mathbb{N} + \cdots + n_k\mathbb{N} \subset \mathbb{N}_{\tilde{W}}$$

holds.

From this property one may derive with the help of elementary number theory that $\mathbb{N}_{\tilde{W}}$ has a rather regular structure: there are an $l_{\tilde{W}}$ and an $m_{\tilde{W}}$ such that for $m \geq m_{\tilde{W}}$ one has $m \in \mathbb{N}_{\tilde{W}}$ iff $m + l_{\tilde{W}} \in \mathbb{N}_{\tilde{W}}$. The "period" $l_{\tilde{W}}$ is just the greatest common divisor of the $n_1, \ldots, n_k$. Since l_0 is a multiple of $l_{\tilde{W}}$ it follows that $m \in \mathbb{N}_{\tilde{W}}$ iff $m + l_0$ for sufficiently large m.

It is now easy to complete the proof. From the definition of $\mathcal{W}$ it follows that

$$R_{x_0}^{\max}(m) = \max_{\tilde{W}}\{G_{\tilde{W}} \mid m \in \mathbb{N}_{\tilde{W}}\},$$

and the observation from the last paragraph implies that for the calculation of $R_{x_0}^{\max}(m)$ one determines the maximum over precisely the same set as for the calculation of $R_{x_0}^{\max}(m + l_0)$. Therefore $R_{x_0}^{\max}(m) = R_{x_0}^{\max}(m + l_0)$ as claimed. □

Corollary 3.2. *In particular one has*

$$\lim_{m\to\infty} \frac{R_{x_0}^{\max}(m)}{m} = \gamma.$$

Thus, for large m, the best possible gain can be approximated by $m\gamma$, and therefore γ is something like "the value of the game".

Remark: We have shown that the sequence $\big(R_{x_0}^{\max}(m)\big)$ is finally l_0-periodic if we define l_0 as the smallest common multiple of the lengths of the cycles in $\mathcal{C}'$. However, in general this l_0 will not be optimal, there might be smaller l such that the sequence is finally l-periodic as well. As simple examples show the best possible l might also depend on the starting position x_0.

4. The general case

In this section the ideas which have been applied successfully in the finite case will be used to deal with arbitrary M. Compactness and the contraction properties of the Γ_ρ will play a crucial role.

Let $\rho_1, \ldots, \rho_l \in \{1, \ldots, r\}$ be arbitrarily given. It follows from lemma 2.1(iv) that

$$\pi_{\rho_1 \cdots \rho_l},\ \pi_{\rho_l \rho_1 \cdots \rho_{l-1}},\ \pi_{\rho_{l-1}\rho_l\rho_1 \cdots \rho_{l-1}},\ \ldots, \pi_{\rho_1 \cdots \rho_l}$$

is a closed walk in the graph associated with our problem: first one has to apply Γ_{ρ_l}, then $\Gamma_{\rho_{l-1}}, \ldots,$ and finally Γ_{ρ_1}. Denote by $\gamma_{\rho_1 \cdots \rho_l}$ the associated stepsize gain, i.e.,

$$\gamma_{\rho_1 \cdots \rho_l} = \frac{1}{l}\big(\varphi_{\rho_l}(\pi_{\rho_1 \cdots \rho_l}) + \varphi_{\rho_{l-1}}(\pi_{\rho_l \rho_1 \cdots \rho_{l-1}}) + \cdots + \varphi_{\rho_1}(\pi_{\rho_2 \cdots \rho_l \rho_1})\big).$$

These numbers are bounded by $\max_{\rho,x} |\varphi_\rho(x)|$, and consequently

$$\gamma := \sup\{\gamma_{\rho_1 \cdots \rho_l} \mid l = 1, 2, \ldots,\ \rho_1, \ldots, \rho_l \in \{1, \ldots, r\}\}$$

is a finite number[5].

Here is the main result of this section:

Proposition 4.1. *If one plays in an optimal way, then the stepsize gain is approximately γ. More precisely: As in the case of finite M one has*

$$\lim_{m\to\infty} \frac{R_{x_0}^{\max}(m)}{m} = \gamma.$$

Remark: It is not true that the sequence $(R_{x_0}^{\max}(m))$ is necessarily finally periodic. However, in many cases one can prove that it is "approximately finally periodic": For every $\varepsilon > 0$ there exist integers m_0, l_0 such that $R_{x_0}^{\max}(m + l_0)$ is ε-close to $R_{x_0}^{\max}(m)$ for $m \geq m_0$.

[5] We note that in the case of finite M one obtains the γ defined in section 3.

The proof of the proposition will be given later. It will be appropriate to introduce a more refined notation first: For $x_0 \in M$ and $\rho_1, \ldots, \rho_k \in \{1, \ldots, r\}$ we will write $x_0^{\rho_1 \cdots \rho_k}$ for $\big(\Gamma_{\rho_k} \circ \cdots \circ \Gamma_{\rho_1}\big)(x_0)$. With this definition the gain $R(x_0; \rho_1, \ldots, \rho_m)$ associated with the choice $\rho_1, \ldots, \rho_m$ is

$$\varphi_{\rho_1}(x_0) + \varphi_{\rho_2}\big(x_0^{\rho_1}\big) + \cdots + \varphi_{\rho_m}\big(x_0^{\rho_1 \cdots \rho_{m-1}}\big).$$

Also one has

Lemma 4.2. *Let $x_0, y_0 \in M$ and $\rho_1, \ldots, \rho_m$ be given.*

(i) *For arbitrary $k \leq m$ the number $R(x_0; \rho_1, \ldots, \rho_m)$ is the sum of $R(x_0; \rho_1, \ldots, \rho_k)$ and $R(x_0^{\rho_1 \cdots \rho_k}; \rho_{k+1}, \ldots, \rho_m)$.*

(ii) *Denote as in section 2 by L and L' the Lipschitz constants associated with the Γ_ρ and the φ_ρ, respectively. Then*

$$|R(x_0; \rho_1, \ldots, \rho_m) - R(y_0; \rho_1, \ldots, \rho_m)| \leq d(x_0, y_0) \frac{L'}{1 - L}.$$

(iii) *Let $\varepsilon > 0$ and suppose that, for certain k, l with $1 \leq k < l \leq m$ one has*

$$d\big(x_0^{\rho_1 \cdots \rho_k}, x_0^{\rho_1 \cdots \rho_l}\big) \leq \varepsilon.$$

Then the following inequality holds:

$$R(x_0; \rho_1, \ldots, \rho_m) \leq (l - k)\gamma + R(x_0; \rho_1, \ldots, \rho_k, \rho_{l+1}, \ldots, \rho_m) + \frac{2\varepsilon L'}{(1 - L)^2}.$$

Proof: (i) is an immediate consequence of the definition, and the idea to prove (ii) has already been sketched above in remark 5 of section 2 .

(iii) The assumption may be rephrased by saying that

$$d\big(\Gamma_{\rho_l} \circ \cdots \circ \Gamma_{\rho_{k+1}}\big(x^{\rho_1 \ldots \rho_k}\big), x^{\rho_1 \ldots \rho_k}\big) \leq \varepsilon$$

so that, by lemma 2.1(i), we know that

$$d\big(x_0^{\rho_1 \ldots \rho_k}, \pi_{\rho_l \ldots \rho_{k+1}}\big) \leq \frac{\varepsilon}{1 - L}.$$

If follows from (ii) that

$$\big|R(x_0^{\rho_1 \ldots \rho_k}; \rho_{k+1}, \ldots, \rho_l) - R(\pi_{\rho_l \ldots \rho_{k+1}}; \rho_{k+1}, \ldots, \rho_l)\big| \leq \frac{\varepsilon L'}{(1 - L)^2}.$$

But $R(\pi_{\rho_l \ldots \rho_{k+1}}; \rho_{k+1}, \ldots, \rho_l) = (l - k)\gamma_{\rho_l \cdots \rho_{k+1}}$, and this number can be estimated by $(l - k)\gamma$.

Note also that (by (ii)) $d\big(x_0^{\rho_1\cdots\rho_k}, x_0^{\rho_1\cdots\rho_l}\big) \le \varepsilon$ implies that

$$\big|R(x_0^{\rho_1\cdots\rho_k};\rho_{l+1},\ldots,\rho_m) - R(x_0^{\rho_1\cdots\rho_l};\rho_{l+1},\ldots,\rho_m)\big| \le \frac{\varepsilon L'}{1-L}$$

and that $R(x_0^{\rho_1\cdots\rho_k};\rho_{l+1},\ldots,\rho_m) = R(x_0;\rho_1,\ldots,\rho_k,\rho_{l+1},\ldots,\rho_m)$
With the help of (i) this leads to

$$\begin{aligned} R(x_0;\rho_1,\ldots,\rho_m) &= R(x_0;\rho_1,\ldots,\rho_k) + \\ &\quad + R(x_0^{\rho_1\cdots\rho_k};\rho_{k+1},\ldots,\rho_l) + R(x_0^{\rho_1\cdots\rho_l};\rho_{l+1},\ldots,\rho_m) \\ &\le (l-k)\gamma + R(x_0;\rho_1,\ldots,\rho_k,\rho_{l+1},\ldots,\rho_m) + \frac{\varepsilon L'}{1-L} + \frac{\varepsilon L'}{(1-L)^2} \\ &\le (l-k)\gamma + R(x_0;\rho_1,\ldots,\rho_k,\rho_{l+1},\ldots,\rho_m) + \frac{2\varepsilon L'}{(1-L)^2} \end{aligned}$$

This completes the proof of the lemma. □

Remark: The proof shows that the inequality in (iii) is even true if the constant γ is replaced by the maximum over the $\gamma_{\rho_1\cdots\rho_{l-k}}$, where the $\rho_1,\ldots,\rho_{l-k}$ run through $\{1,\ldots,r\}$.

We now are prepared for the proof of proposition 4.1. First we show that

$$\limsup_{m\to\infty} \frac{R_{x_0}^{\max}(m)}{m} \le \gamma.$$

To this end, fix x_0 and let $\varepsilon > 0$ be given. Since M is compact, there is an integer n_0 with the following property: whenever one considers a finite number of points $x_1,\ldots,x_{n_0} \in M$ there are i,j with $i \ne j$ such that $d(x_i,x_j) \le \varepsilon$. Let arbitrary $\rho_1,\ldots,\rho_m$ be given, where $m > n_0$. As usual we put $x_0^{\rho_1} := \Gamma_{\rho_1}(x_0)$, $x_0^{\rho_1\rho_2} := \Gamma_{\rho_2}(x_0^{\rho_1})$, … By the preceding remark we may choose $k < l$ such that $d(x_0^{\rho_1\ldots\rho_k}, x_0^{\rho_1\ldots\rho_l}) \le \varepsilon$, we also can arrange it that the length of the sequence $x_0, x_0^{\rho_1}\ldots, x_0^{\rho_1\ldots\rho_k} x_0^{\rho_1\ldots\rho_{l+1}},\ldots, x_0^{\rho_1\ldots\rho_m}$ is bounded by n_0.
Now the lemma comes into play, it implies that

$$\begin{aligned} R(x_0;\rho_1,\ldots,\rho_m) &\le (l-k)\gamma + R(x_0;\rho_1,\ldots,\rho_k,\rho_{l+1}\cdots\rho_m) + 2\varepsilon\frac{L'}{(1-L)^2} \\ &\le (l-k)\gamma + n_0\max_\rho ||\varphi_\rho|| + 2\varepsilon\frac{L'}{(1-L)^2}. \end{aligned}$$

Thus

$$\frac{R(x_0;\rho_1,\ldots,\rho_m)}{m} \le \gamma + \frac{m-(l-k)}{m}\gamma + \frac{n_0}{m}\big(\max_\rho ||\varphi_\rho|| + 2\varepsilon\frac{L'}{(1-L)^2}\big),$$

where $m-(l-k) \leq n_0$. The number n_0 can be chosen independent of the particular $\rho_1, \ldots, \rho_m$ so that also

$$\frac{R_{x_0}^{\max}(m)}{m} \leq \gamma + \frac{m-(l-k)}{m}\gamma + \frac{n_0}{m}\big(\max_\rho ||\varphi_\rho|| + 2\varepsilon\frac{L'}{(1-L)^2}\big)$$

must hold. It follows immediately that

$$\limsup \frac{R_{x_0}^{\max}(m)}{m} \leq \gamma.$$

It remains to show that $\liminf R(x_0;\rho_1,\ldots,\rho_m)/m \geq \gamma$ also holds. Let x_0 and $\varepsilon > 0$ be given. Choose a cycle $\rho'_1, \ldots, \rho'_l$ such that the stepsize gain $\gamma_{\rho'_1\cdots\rho'_l}$ satisfies $\gamma_{\rho'_1\cdots\rho'_l} \geq \gamma - \varepsilon$.

We will find a walk which provides nearly the optimal stepsize gain by moving first from x_0 "close to" $\pi_{\rho'_l\cdots\rho'_1}$ by repeatedly applying the maps $\Gamma_{\rho'_l}, \ldots, \Gamma_{\rho'_1}, \Gamma_{\rho'_l}, \ldots, \Gamma_{\rho'_1}, \ldots$ (cf. lemma 2.2.(iii)). In this way we obtain suitable $\rho_1, \ldots, \rho_{m_0}$ such that

$$d(\pi_{\rho'_1\cdots\rho'_l}, x_0^{\rho_1,\ldots,\rho_{m_0}}) \leq \varepsilon.$$

The walk we are looking for is now defined as follows: It starts with $\rho_1, \ldots, \rho_{m_0}$ and then one chooses again and again the sequence $\rho'_l, \ldots, \rho'_1$. Suppose that the walk has length m, we denote the ρ-values by $\rho_1, \ldots, \rho_m$. The gain associated with the first m_0 steps is bounded from below by $-m_0 \max_\rho ||\varphi_\rho||$. Then follows a number of complete cycles (say k rounds) which contribute with a gain of at least $kl(\gamma-\varepsilon)$. There possibly remain some steps (at most $(l-1)$) since the last cycle is not completed, the associated gain is at least $-(l-1)\max_\rho ||\varphi_\rho||$. Our argument implies that kl, the number of steps in the complete cycle, is at least $m - m_0 - l + 1$, and in this way we arrive at the following inequality:

$$R(x_0;\rho_1,\ldots,\rho_m) \geq m(\gamma-\varepsilon) - C,$$

where C can be chosen independent of m. It follows that

$$\frac{R^{\max}(m)}{m} \geq \frac{R(x_0;\rho_1,\ldots,\rho_m)}{m} \geq \gamma - \varepsilon - \frac{C}{m},$$

and therefore

$$\liminf \frac{R^{\max}(m)}{m} \geq \gamma - \varepsilon$$

must hold. This completes the proof of the proposition.

5. The calculation of the constant γ

We have seen that with $M, \Gamma_\rho, \varphi_\rho$ $(\rho = 1, \ldots, r)$ there is associated a characteristic constant γ which plays the role of "the value of the game". There seems to be no simple way to determine this number. The next proposition shows that an approximate calculation is feasable if the *fractal dimension* of the fractal F associated with the situation is "very small"[6]:

Proposition 5.1. *Let $\varepsilon > 0$ be given and suppose that an integer n has the property that for each choice of n points $y_1, \ldots, y_n \in F$ there are y_i, y_j with $i \neq j$ such that $d(y_i, y_j) \leq \varepsilon$. Then*

$$\left| \gamma - \max_{\rho_1, \ldots, \rho_n} \gamma_{\rho_1 \cdots \rho_n} \right| \leq \frac{2\varepsilon L'}{(1-L)^2}.$$

i.e., for an approximate calculation it is not necessary to consider all *cycles but "only" r^n of them.*

Remark: Denote by N_ε the best possible n for a given positive ε and recall that the fractal dimension δ_F of F is the limit of the numbers $-\log N_\varepsilon / \log \varepsilon$ when ε tends to zero (provided that this limit exists); cf. [2]. We thus can restate the previous assertion by saying that the complexity to calculate γ up to an error of order ε is at most of the order $r^{1/\varepsilon^{\delta_F}}$.

Proof: Define $\gamma_n := \max_{\rho_1, \ldots, \rho_n} \gamma_{\rho_1 \cdots \rho_n}$ and consider a cycle defined by $\rho_1, \ldots, \rho_m$ of arbitrary length:

$$\pi_{\rho_1 \ldots \rho_m}, \ \pi_{\rho_m \rho_1 \ldots \rho_{m-1}}, \ \pi_{\rho_{m-1} \rho_m \rho_1 \ldots \rho_{m-2}}, \ \ldots, \pi_{\rho_1 \ldots \rho_m}.$$

We will show that $\gamma_{\rho_1 \ldots \rho_m} \leq \gamma_n + 2\varepsilon L'/(1-L)^2$. This implies that $\gamma \leq \gamma_n + 2\varepsilon L'/(1-L)^2$, and the assertion follows since it is true by definition that $\gamma_n \leq \gamma$.

Let us denote the starting point $\pi_{\rho_1 \ldots \rho_m}$ of our cycle by z_0, the next by z_1, etc. Then

$$R(z_0; \rho_m, \ldots, \rho_1) = m\gamma_{\rho_1 \ldots \rho_m}$$

by definition. If we suppose that m is bigger than n it follows from our assumption that we can find z_k, z_l with $k < l \leq k+n$ such that $d(x_k, x_k) \leq \varepsilon$. Now, by lemma 4.2 and the remark following the proof of it, we have

$$R(z_0; \rho_m, \ldots, \rho_1) \leq (l-k)\gamma_n + R(z_0; \rho_m, \ldots, \rho_{l+1}, \rho_k, \ldots, \rho_1) + \frac{2\varepsilon L'}{(1-L)^2}.$$

[6] For the definition of F cf. lemma 2.1.

Similarly we treat $R(z_0; \rho_m, \dots, \rho_{l+1}, \rho_k, \dots, \rho_1)$: we replace a subwalk of length bounded by n by a walk in a cycle, and this causes an error of at most γ_n times the length of the cycle plus $2\varepsilon L'/(1-L)^2$. This reduction will be done again and again, after at most m steps we arrive at

$$\begin{aligned} m\gamma_{\rho_1 \cdots \rho_m} &= R(z_0; \rho_m, \dots, \rho_1) \\ &\leq m\gamma_n + m\frac{2\varepsilon L'}{(1-L)^2}. \end{aligned}$$

It remains only to divide by m to complete the proof. □

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Fachbereich Mathematik und Informatik, Freie Universität Berlin, Arnimallee 2–6, D-14 195 Berlin, Germany; e-mail: behrends@math.fu-berlin.de