Sequences with nontrivial Sums: Algebra meets Magic

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Let us start this article with the desription of a magic trick. The magician presents some cards. They are shuffled with the help of the audience and seven of them are selected. These are cards with values $2, 3, 4, 5, 6, 7$ and an ace that counts 1. The magician puts them in the selected order face-down as a circle on the table, and a spectator is invited to perform a walk on this circle. First she chooses one of the cards and turns it face up:

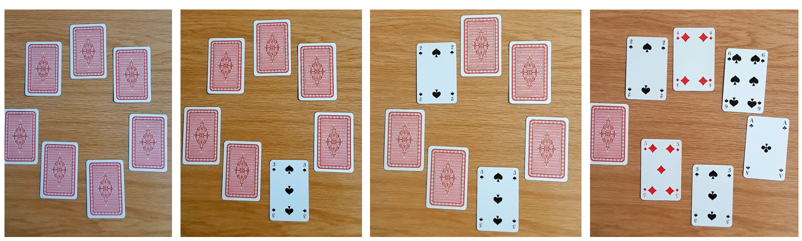

Figure 1: A “walk” on a circle of seven cards.

She walks as many steps as the number on the card shows (3 steps in our example) clockwise in the circle. The card where she arrives is switched and the procedure is repeated: see the figure for some steps of the walk.

Finally she has visited the cards $3 \to 2 \to 6 \to 4 \to 5 \to 1$ and from there the next step would lead to the first card.

But what about the remaining card that is still face down? The magician switches it and it turns out that it is the only card that lies face up in another deck that was on the table from the beginning. How could this be after these extensive shuffles and random choices?

The motivation of the present article was to understand the mathematical background of this magic trick. Why seven cards? Why this special order? ...

In the first section we introduce “sequences with nontrivial sums” in $\mathbb{Z}_n$, a definition that will be crucial for our investigations. The next section contains the result that such sequences always exist if n is odd. Then we resume the discussion of the magic trick described above, and the article closes with some remarks.

We note that in [3] – [11] one finds other articles of the author where facts from various areas of mathematics are used to present magic tricks. [1] is a collection with rather elementary contributions written for a general public, and [2] contains 15 chapters written for mathematicians; short versions of some of the articles [3] – [11] are also included ln this book.

Sequences with nontrivial sums

Let n be an integer, $n > 2$. Suppose that we have a sequence of distinct nonzero elements $(a_1, \ldots, a_{n-1})$ in the ring $\mathbb{Z}_n$, i.e., a permutation of $1, \ldots, n-1$. We say that $(a_1, \ldots, a_{n-1})$ has *nontrivial sums* if the following holds: Whenever one chooses at most $n-2$ consecutive terms, $a_k, a_{k+1}, \ldots, a_{k'}$, then $\sum_{i=k}^{k'} a_i \neq 0$. In view of the origin of the motivation to study such sequences we will be interested only in the case when $\sum_{i=1}^{n-1} a_i = 0$. (Cf. section 3). This happens iff n is odd, and in the sequel we will assume that $n = 2m + 1$.

For example, in $\mathbb{Z}_5$, the sequence $(1, 2, 4, 3)_5$ has nontrivial sums, but $(1, 2, 3, 4)_5$ fails to have this property since $2 + 3 = 0$. (The subindex 5 indicates that the sequence is in $\mathbb{Z}_5$. In this way one could also write $(1, 2, -1, -2)_5$ for the first example, a notation that would be ambiguous without the subindex.)

Let us collect some easy first results:

1 Lemma: *Let n be an odd number and suppose that $(a_1, \ldots, a_{n-1})$ is a sequence of distinct elements in $\{1, 2, \ldots, n-1\}$. If $(a_1, \ldots, a_{n-1})$ has nontrivial sums then so do also*

- *the reversed sequence $(a_{n-1}, a_{n-2}, \ldots, a_1)$,*
- *the multiples $(a \cdot a_1, a \cdot a_2, \ldots, a \cdot a_{n-1})$ for invertible a*
- *and the translations $(a_k, a_{k+1}, \ldots, a_{n-1}, a_1, \ldots, a_{k-1})$ for all k.*

Proof: The first two assertions are obviously true. For the third we use the fact that $\sum_i a_i = 0$. It follows that $\sum_{i=k}^{k'} a_i = -\sum_{i \notin \{k,\ldots,k'\}} a_i$, and this easily implies that all translations have nontrivial sums. □

We will prove in *the next section* that a sequence with nontrivial sums exists for every odd n. The proof will be elementary but somehow involved. Next, we indicate how this fact (applied for rather small n) can be used to perform a *magic trick*. And in a *final section* one finds some concluding remarks.

In the theory of *combinatorial designs* one considers similar questions (see [13] and the references cited there). There the *Alspach conjecture* plays an important role: Let A be a subset of any $\mathbb{Z}_n \setminus \{0\}$ with k elements such that $\sum_{a \in A} a \neq 0$. Then one can arrange the elements of A as $a_1, \ldots, a_k$ such that $\sum_{i=s}^{i=s'} a_i \neq 0$ for all s, s' with $1 \leq s \leq s' \leq k$. The Alspach conjecture is still open.

Many partial results are known. In particular it is true for even n that $A = \{1, \ldots, n-1\}$ can be reordered as $(a_1, \ldots, a_{n-1})$ such that all $\sum_{i=s}^{i=s'} a_i$ are different from zero. (See [12], theorem 1.)

The proof: Let n be even. We claim that $(1, -2, 3, -4, \ldots, 2, -1)_n$ is a permutation of the set $\mathbb{Z}_n \setminus \{0\}$ with the desired properties. It is visualized in figure 1 for the case $n = 10$ by the red zig-zag line,

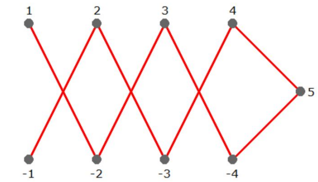

Figure 2: A visualization of the $a's$ for $n = 10$.

Put $S_k := a_1 + \cdots + a_k$ for $k = 1, \ldots, n-1$. In our example $n = 10$ one obtains

k	1	2	3	4	5	6	7	8	9
S_k	1	-1	2	-2	3	-3	4	-4	5

And in general, if $n = 2m$, one has

$$\{S_k \mid k = 1, \ldots, n-1\} = \{\pm 1, \pm 2, \ldots, \pm(m-1), m\}$$

as an easy consequence of the fact that $a_k = k$ for odd k and $a_k = -k$ for even k. It follows that the S_k are pairwise different and this proves the claim.

The case $A = \{1, \ldots, n-1\}$ of the Alspach conjecture is in a sense complementary to the problem investigated here since *we* study *odd* n whereas in the theory of combinatorial designs only the *even* n are of interest.

Sequences with nontrivial sums always exist if n is odd

2 Proposition: *Every odd n admits a sequence with nontrivial sums.*

Proof: Let $n = 2m+1$ be given. We define a permutation of $\{1, 2, \ldots, n-1\}$ as follows: $a_1 := 1$, $a_i := (-1)^i \, 2(i-1)$ for $i = 1, \ldots, m$, $a_{m+1} := -1$ and $a_{m+1+i} := -a_{m+1-i}$. Note that the a_i really exhaust $\{1, \ldots, n-1\}$: The fact that

$$\{1, \ldots, n-1\} = \{1, -1\} \cup \{2i \mid i = 1, \ldots, m-1\} \cup \{-2i \mid i = 1, \ldots, m-1\}$$

holds is a consequence of the invertibility of 2 in $\mathbb{Z}_n$.

As an example consider the case $n = 11$. Here the a_i are defined by

$$(a_1, \ldots, a_{10}) = (1, 2, -4, 6, -8, -1, 8, -6, 4, -2)_{11} = (1, 2, 7, 6, 3, 10, 8, 5, 4, 9)_{11}.$$

It will be helpful to visualize this sequence (see figure 2). The permutation is defined by the red zig zag line that oscillates between $\{2i \mid i = 1, \ldots, m\}$ and $\{-2i \mid i = 1, \ldots, m\}$:

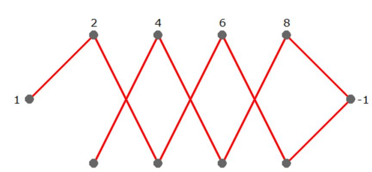

Figure 3: A visualization of the $a's$ for $n = 11$.

It remains to show that we have defined a sequence with nontrivial sums, i.e., that $S_{k,k'} := a_k + \cdots + a_{k'} \neq 0$ holds for $1 \leq k \leq k' \leq n-1$, if $(k, k') \neq (1, n-1)$.

A special case will play a crucial role. As a preparation consider the $S_{k,k'}$ for $2 \leq k \leq k' \leq 5$ when $n = 11$ (see figure 2):

$$S_{2,2} = 2; S_{2,3} = -2; S_{2,4} = 4; S_{2,5} = -4; S_{3,3} = -4;$$

$$S_{3,4} = 2; S_{3,5} = -6; S_{4,4} = 6; S_{4,5} = -2; S_{5,5} = -8.$$

They all are not in $\{-1, 0, 1\}$, and this is always the case:

Claim: $S_{k,k'} \in A := \{2i \mid i = 1, \ldots, m\} \cup \{-2i \mid i = 1, \ldots, m\}$ for $2 \leq k \leq k' \leq m$. In particular, $S_{k,k'} \notin \{-1, 0, 1\}$.

Proof of the claim: Let $k \in \{2, \ldots, m\}$ and $k' = k + t$ with $k + t \leq m$ be given. Then

$$\begin{aligned} S_{k,k+t} &= a_k + \cdots + a_{k+t} \\ &= 2(-1)^k[(k-1) - k + (k+1) - (k+2) + \cdots \pm (k+t-1)]. \end{aligned}$$

If $t = 2s+1$ is odd, the resulting number is $2(-1)^k[-1-0+1-2+3-\cdots\pm(t-1)]$; here $-1-0+1-2+3-\cdots\pm(t-1)$ equals $-s-1$ so that $S_{k,k+t} = 2(-1)^k(-s-1)$, and in the case of even $t = 2s$ one obtains $2(-1)^k[k-1+1+\cdots+1] = 2(-1)^k(k+s-1)$. Since $-s-1$ and $k+s-1$ lie in $\{-(m-1), \ldots, -2, -1, 1, 2, \ldots, (m-1)\}$ the claim follows.

End of the proof of the claim.

In this way we have in particular settled

Case 1: $S_{k,k'} \neq 0$ for $2 \leq k \leq k' \leq m$.

The remaining cases follow easily from the claim:

Case 2: $S_{1,k'} \neq 0$ for $2 \leq k' \leq m$.

This follows from $S_{1,k'} = 1 + S_{2,k'}$. By the claim $S_{2,k'} \in A$, and $-1 \notin A$.

Case 3: $S_{1,m+1} \neq 0$.

$S_{1,m+1} = S_{2,m}$ since $a_{m+1} = -a_1$.

Case 4: $S_{1,m+1+k} \neq 0$.

In this sum there occur summands $a, -a$. It is equal to $S_{2,m-k-1}$.

Similarly all other situations are treated. Note, e.g., that $S_{m+1+k,m+1+k'} = -S_{m+1-k',m+1-k}$ for $1 \leq k \leq k' \leq m-1$ since $a_{m+1+i} = -a_{m+1-i}$ for $i = 1, \ldots, m-1$. □

A magic trick associated with sequences with nontrivial sums

Recall that in the trick presented at the beginning the spectator has visited the cards $3 \to 2 \to 6 \to 4 \to 5 \to 1$ and from there the next step would lead to the first card.

Note that these numbers have a *surprising property:* The spectator started at the 3, but a start at another number would have produced (a translation of) the same walk, e.g. $4 \to 5 \to 1 \to 3 \to 2 \to 6 \to 4$ when starting at 4.

Only one card was not turned face up: It is the 7, and a "walk" starting there would be rather boring: $7 \to 7 \to 7 \cdots$.

Phrased in mathematical terms the situation is as follows. We have the permutation $(b_0, \ldots, b_6) = (7, 2, 4, 6, 1, 3, 5)$ of $\mathbb{Z}_7$ and these numbers give rise to walks: Always step from i to $i+b_i$. Then there will be two types of walks: One that never leaves $\{7\}(=\{0\})$ and another that visits all points in $\{1, 2, 3, 4, 5, 6\}$, regardless where the walk starts.

The natural generalization to arbitrary n reads as follows:

3 Definition: *Let n be odd and $(b_0, \ldots, b_{n-1})$ a permutation of $\mathbb{Z}_n$. We say that $(b_0, \ldots, b_{n-1})$ has the* long walk property (lwp for short)*, if the following holds: If one always steps from i to $i + b_i$ then there are two types of walks: one that never leaves $\{i_0\}$ (where $b_{i_0} = 0$) and another that visits all points in $\{i \mid i \neq i_0\}$ whenever one starts the walk in this subset.*

Just before the preceding definition we introduced $(0, 2, 4, 6, 1, 3, 5)_7$, a sequence with lwp. And here is an example in Z_{11}: $(0, 1, 2, 9, 5, 3, 4, 7, 10, 8, 6)_{11}$. As a counterexample consider $(0, 4, 3, 2, 1)_5$ in $\mathbb{Z}_5$: all walks end in $\{0\}$.

In the next proposition we show that lwp sequences and sequences with nontrivial sums are closely related:

4 Proposition: *Let n be odd.*

(i) With $(b_0, \ldots, b_{n-1})$ also the translations $(b_k, b_{k+1}, \ldots, b_{n-1}, b_0, \ldots, b_{k-1})$ are lwp sequences for all k.

(ii) Every sequence $(a_1, \ldots, a_{n-1})$ with nontrivial sums generates an lwp sequence.

(iii) Every lwp sequence $(b_0, b_1, \ldots, b_{n-1})$ generates a sequence with nontrivial sums.

Proof: (i) This fact is obvious.

(ii) Let $(a_1, \ldots, a_{n-1})$ have nontrivial sums. We define the b_i as follows. $b_0 := a_1$, $b_{a_1} := a_2$, then $b_{a_1+a_2} := a_3$, and generally $b_{a_1+\cdots+a_r} := a_{r+1}$ for $r = 0, \ldots, n-2$. Since the $a's$ have nontrivial sums the $n-1$ points $0, a_1, a_1 + a_2, \ldots, a_1 + \cdots + a_{n-2}$ where we have defined the b_i are distinct so that we have fixed the b_i for $n-2$ elements in $\mathbb{Z}_n$. Consequently precisely one i_0 is missing, we put $b_{i_0} := 0$.

It is routine to show that $(b_0, \ldots, b_{n-1})$ has lwp: for example a walk that starts at an $i \neq i_0$ will be back at i after $n-1$ steps since $\sum_i a_i = 0$, and that there are no shorter walks is due to the fact that the $a's$ have nontrivial sums.

As an example consider $(2, 5, 3, 4, 1, 7, 8, 6)_9$ in $\mathbb{Z}_9$. This sequence generates the lwp sequence $(2, 4, 5, 6, 8, 1, 7, 3, 0)_9$.

(iii) Now we start with an lwp-sequence $(b_0, \ldots, b_{n-1})$ in $\mathbb{Z}_n$. Choose any i_1 such that $b_{i_1} \neq 0$; we know that the walk that starts at i_1 meets $n-1$ points.

Put $i_{k+1} := i_k + b_{i_k}$ for $k := 1, \ldots, n-2$ (these are the points that are visited by the walk) and $a_k := b_{i_k}$ for $k = 1, \ldots, n-1$.

The $(a_1, \ldots, a_{n-1})$ have the following properties:

- They are different from zero. (Because the walk starting in i_1 stays in the set $\{i \mid b_i \neq 0\}$.)
- They are distinct. (Because the walk visits all elements of the subset $\{i \mid b_i \neq 0\}$ only once.)
- They have nontrivial sums. (Because all walks starting in $\{i \mid b_i \neq 0\}$ are closed with length $n-1$.)

As an illustration consider $(b_0, \ldots, b_8) = (0, 2, 3, 5, 7, 1, 4, 6, 8)_9$ and $i_1 := 1$. Then $(a_1, \ldots, a_8) = (2, 5, 8, 6, 7, 3, 1, 4)_9$. □

Note: Lwp sequences fail to have the same permanence properties as sequences with nontrivial sums: $(0, 1, 2, 3, 4)_5$ has lwp, but the sequence $3 \cdot (0, 1, 2, 3, 4)_5 = (0, 3, 1, 4, 2)_5$ and also the reversed sequence $(4, 3, 2, 1, 0)_5$ have not.

After these preparations we can desribe a *mathematical magic trick* that generalizes the one from the beginning. We first present the *raw version* of the trick we have in mind, later we will propose several *refinements*.

Step 1: Choose any odd integer n, any sequence $(a_1, \ldots, a_{n-1})_n$ with nontrivial sums and generate an lwp sequence $(b_0, b_1, \ldots, b_{n-1})_n$. Here we will illustrate the trick with $n = 7$, $(a_1, \ldots, a_6)_7 = (2, 6, 4, 5, 1, 3)_7$ and the lwp sequence $(b_0, \ldots, b_6)_7 = (0, 2, 4, 6, 1, 3, 5)_7$

Step 2: Prepare cards that, when held face down, show on their back the numbers $b_0, b_1, \ldots, b_{n-1}$ (in that order from top to botton). With blanko cards marked with these numbers this is possible for every n, but with playing cards n should be at most 13: Then one could define the values of Ace, Jack, Queen, King as $1, 11, 12, 13 (= 0)$, respectively.

Prepare also something that could be used later to prove that you knew in advance that the number $n (= 0)$ would play a particular role: E.g., if you work with $n = 7$ and if you have represented the 7 by the seven of diamonds then write "seven of diamonds" on a sheet of paper and put it in an envelope that is placed on the table. Or prepare a seven of diamonds in another deck of cards as the only card that is face up among other cards that lie all face down.

Step 3: Place the $(b_0, \ldots, b_{n-1})$-deck face down on the table. The audience might suspect that the cards are prepared. (In fact they are!) In order to convince them that this is not the case you allow several cuts or even operations that look more complicated but also amount only in cutting the deck (like false cuts or Charlier shuffle; in the internet these terms are explained). After this the order of the deck might have changed to a translation of the original order.

Step 4: Deal the cards one by one clockwise as a circle of n cards. A spectator might point to any of these cards. It is switched face up.
Case 1: It is the card that represents $n (= 0)$. You exclaim: "Believe it or not! I knew in advance that you would choose precisely this card!" Then prove this by showing your prediction in the envelope or the card that is the only switched one in the other deck.

Case 2: Another card is chosen. This gives rise to a walk as described at the beginning of this section: Walk clockwise as many steps as the number on the switched card shows. Switch this card. Continue walking. Finally there will be $n-1$ cards that lie face up. And you ask: "Which one might be that card that resisted to be be laid face up?" You switch it, and everyone can see that it is announced in the prepared envelope (or that it coincides with the only card in the other deck that shows face up).

Refinements: It has to be amphasized that this is only a raw version. Your creativity is asked in order to convince the audience that here something that is really magic happens. The distinguished card survived the switching process, even after after the order of the cards was disturbed. Etc.

Here are some concrete proposals for refinements.

1. Let us illustrate what we mean with the simple sequence $(1,2,3,4,5,6,7)$. If you double it, you obtain $(1,2,3,4,5,6,7,1,2,3,4,5,6,7)$, and this sequence is 7-*periodic* in the following sense: For every element in this sequence there is the same element if you walk seven steps to the right. This is meant *cyclically*: If 7 steps to the right are not possible, you start again from the beginning.

And 7-periodic sequences have *the following properties*.

- Every translation by k elements, e.g. $(4,5,6,7,1,2,3,4,5,6,1,2,3)$ in the case $k=3$, is again 7-periodic.

- If you remove cards from the left and from the right, in total 7 cards, then the remaining sequence is a translation of the original one with 7 elements.

  An example: Remove 4 from the left and 3 from the right, there remain $5,6,7,1,2,3,4$, a translation of $(1,2,3,4,5,6,7)$.

These facts are obviously true, and it should be clear that one could replace 7 by any n and the simple order of the cards by any other.

And here is the important consequence for mathematical magic: Suppose that you want to start your magic trick with a pack of n cards in a certain order $(a_0, a_1, \ldots, a_{n-1})$ and that any translation of this order would also work well.
Then you could start your trick with $2n$ cards in the order

$$a_0, a_1, \ldots, a_{n-1}, a_0, a_1, \ldots, a_{n-1},$$

this deck can be cut several times by the spectators, one of them can remove n cards from top and bottom, and further cuts might follow. You can be sure that your original order survived, maybe it will be a translation of it. The trick can begin!

As an example suppose that we want to work with 7 cards and the lwp-sequence $(4,6,1,3,5,7,2)_7$. Then prepare a deck like the first one in figure 4.

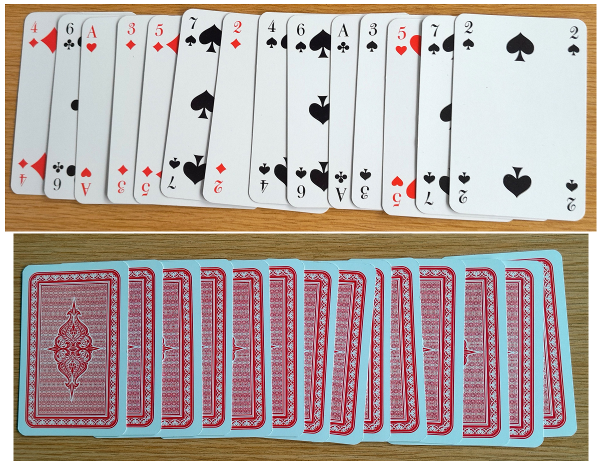

Figure 4: A "walk" on a circle of seven cards.

The audience, however, sees the cards as in the second figure in figure 4; now the 4 of diamonds is the card on the left hand side. (It is possible to turn the deck face up, to fan it out and to show it shortly. But then you should take care that nobody can see that there are two identical cards that show the seven of spades: One of them will later be the distinguishd card, and the seven of spades is also the switched card in the extra deck.)

Now seven cards can be removed from left and/or right, some cuts are allowed, and then the presentation can begin.

2. In $\mathbb{Z}_n$ one has $k = k + n$ for every k. Thus, if you represent numbers by cards, it is not necessary always to use k when this number is relevant at this position. E.g., if you work with $n = 7$, you can replace every now and then an Ace (resp. a 2 resp. a 3) by an 8 (resp. a 9 resp. a 10).

3. You can "personalize" the trick. In our example where the seven of spades is the remainig card you could write something on it: "Happy birthday, dear X." if your presentation is at the occasion of X's birthday, etc. Needless to say that this must not be visible when you fan out the cards.

Important note: The order of the cards is crucial. If you work with the lwp sequence $(2, 4, 5, 6, 8, 1, 7, 3, 0)_9$ and the deck is shown face down, then it is absolutely necessary that the next card after the 2 is the 4. If they would lie the other way round the trick would not work. Also it is important that everything happens *clockwise*: the way to deal the cards to obtain a circle and the walks of the spectators. (In fact, both actions could be changed to "counterclockwise", but "clockwise" and "counterclockwise" must not be mixed.)

Concluding remarks

1. Some time ago a member of my magic circle presented a trick with seven cards that he found in the book "Fast von selbst" by Werner Miller (see [14]). The trick works since $(0, 2, 4, 6, 1, 3, 5)$ has lwp, and it was natural to ask whether

there are other lwp sequences and whether one could present a similar trick with another number of cards. As already mentioned it were these questions that motivated the present investigations.

2. Why did we restrict ourselves to odd n? The answer: We had in mind to apply our results to a magic trick. This makes it necessary to generate from $(a_1, \ldots, a_{n-1})$ an lwp sequence $(b_0, \ldots, b_{n-1})$, and this only works if $\sum_{i=1}^{n-1} i = 0$, i.e., if n is odd.

3. If one tries to find $(a_1, \ldots, a_{n-1})$ in $\mathbb{Z}_n$ with nontrivial sums one could proceed as follows:
– Start with any $a_1 \neq 0$.
– If you have already constructed $a_1, \ldots, a_k$ find the next one, a_{k+1}, as follows: In order to have the nontrivial sum condition satisfied one has to guarantee that a_{k+1} is different from all $a_1, \ldots, a_k$, and $a_j + \cdots + a_k + a_{k+1} \neq 0$ for all $j = 1, \ldots, k$. To state it otherwise: a_{k+1} must not be in

$$T_{a_1,\ldots,a_k} := \{a_1, \ldots, a_k\} \cup \{-(a_j + \cdots + a_k) \mid j = 1, \ldots, k\}.$$

This will be easy for small k, but nearly always it happens that the "taboo set" $T_{a_1,\ldots,a_k}$ is all of $\{1, \ldots, n-1\}$ for some $k < n-2$: Note that it contains at least $(a_1, \ldots, a_k)$ and possibly k further elements so that it tends to be "too big" when $k > n/2$. Then this approach leads to a dead end.

The main idea in the proof of proposition 2 is to choose the a_i such that $T_{a_1,\ldots,a_k}$ grows slowly when k approaches $n/2$, and it helps when these numbers contain as may pairs as possible of the form $a, -a$. In fact it were extensive computer experiments that led to the construction in the proof of the proposition.

4. Computer experiments show that there are sequences with nontrivial sums that are not of the form of the example in proposition 2 or derived from this example by inversion, translation or multiplication with an invertible element. In fact, for $n \leq 7$ all examples have this form, but already for $n = 9$ there are further sequences with nontrivial sums. (E.g. $(1,3,7,5,8,6,2,4)_9$; this sequence is not constructed from $(1,2,-4,6,-1,-6,4,-2)_9$ since there are no a_i such that $a_{i+1} = -a_{i-1}$.)

Nevertheless, the proportion of sequences with nontrivial sums among the permutations of $1, 2, , \ldots, n-1$ is more and more tiny with increasing n: $\approx 2/100$ for $n = 7$; $\approx 2/1000$ for $n = 9$; $\approx 2/10000$ for $n = 11$, ...

5. It would be interesting to find other recipes for finding sequences with nontrivial sums than that of the proof of proposition 2.

6. If one weakens the requirement that the $(a_1, \ldots, a_{n-1})$ exhaust $\{1, \ldots, n-1\}$ then it is rather easy to find examples with $\sum_{i=k}^{k'} a_i \neq 0$ for $(k, k') \neq (1, n-1)$ and $\sum_{i=1}^{n-1} a_i = 0$, also for even n. One simply can take $(1, 1, \ldots, 1, 1, 2)_n$ (with the lwp sequence $(0, 1, \ldots, 1, 2)_n$), but also more complicated examples can easily be found, for example $(4, 3, 2, 5, 4)_6$ (with $0, 4, 2, 4, 5, 3$ as the associated lwp sequence). Such sequences can also be used for our magic trick.

Acknowledgement: I am grateful to Günter M. Ziegler for providing the reference [13]. There I learned that, surprisingly, there are connections between sequences that have their origin in a magical trick and combinatorial designs.

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