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Where is matrix multiplication locally open?

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ABSTRACT

Let (M_1, d_1), (M_2, d_2) be metric spaces. A map $f : M_1 \to M_2$ is said to be *locally open* at an $x_1 \in M_1$, if for every $\varepsilon > 0$ one finds a $\delta > 0$ such that $B\big(f(x_1), \delta\big) \subset f\big(B(x_1, \varepsilon)\big)$; here $B(x, r)$ stands for the closed ball with center x and radius r. We are particularly interested in the following special case: X, Y, Z are normed spaces, the spaces $L(X, Y)$, $L(Y, Z)$, $L(X, Z)$ of linear continuous operators are provided with the operator norm, and the map under consideration is the bilinear map $(S, T) \mapsto S \circ T$ (from $L(Y, Z) \times (L(X, Y)$ to $L(X, Z))$. For which pairs $(S_0, T_0) \in L(Y, Z) \times (L(X, Y)$ is it locally open?

The main result of the paper gives a complete characterization of pairs (S, T) at which this map is locally open in the case of finite-dimensional spaces X, Y, Z.

1. The problem

Let $\mathcal{A}$ be a normed algebra and $(x_0, y_0) \in \mathcal{A}$. We say that *multiplication is locally open* at (x_0, y_0) if for every $\varepsilon > 0$ one can find a $\delta > 0$ such that

$$\{x_0 \cdot y_0 + w \mid ||w|| \leq \delta\} \subset \{(x_0 + x) \cdot (y_0 + y) \mid ||x||, ||y|| \leq \varepsilon\}.$$

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The case of commutative $\mathcal{A}$ has been investigated in several papers (see [1–7]). Here we aim at characterizing the pairs (x_0, y_0) where multiplication is locally open in the noncommutative normed algebra $\mathcal{M}_n$ of real or complex $n \times n$-matrices for arbitrary $n \in \mathbb{N}$.

In the framework of singularity theory the problem whether certain nonlinear maps are locally open at a given point has been studied for many decades. However, the interest in the special case of "natural" bilinear maps that occur in functional analysis is relatively new.

Whereas the "commutative" case (like pointwise multiplication in spaces of measurables or continuous functions) leads to problems in measure theory and topology the noncommutative matrix multiplication corresponds to a nonlinear map from $2n^2$- to n^2-dimensional space, and there seem to be no general results that could be used here unless this mapping has full rank. We will show that rather elementary linear algebra leads to a complete characterization.

Another viewpoint is also possible. Let X, Y, Z be normed linear spaces and denote by $L(X, Y)$, $L(Y, Z)$ and $L(X, Z)$ the spaces of continuous linear operators from X to Y etc. (These spaces are provided with the operator norm.) One can consider the bilinear map $\Phi : L(Y, Z) \times L(X, Y) \to L(X, Z)$, $(S, T) \mapsto S \circ T$, and one may ask: at which pairs (S_0, T_0) is Φ, the operator "multiplication", locally open? The preceding case of normed algebras covers the situation $X = Y = Z$, but what can be said in general? We provide a complete characterization for the case of finite dimensional X, Y, Z in section 2, and section 3 is concerned with the special case of multiplication in $\mathcal{M}_n$.

2. Where is $(S, T) \mapsto S \circ T$ locally open?

First we collect some facts that concern arbitrary normed spaces X, Y, Z over the scalar field $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}\}$. (We will drop the symbol "$\circ$" for the composition of maps.)

Lemma 2.1. *Let $S_0 \in L(Y, Z)$ and $T_0 \in L(X, Y)$ be given.*

(i) *Suppose that T_0 has the following property: for every $\varepsilon > 0$ there exists $T_\varepsilon \in L(X, Y)$ with $||T_\varepsilon|| \le \varepsilon$ such that $S_0 T_\varepsilon = 0$, and $T_0 + T_\varepsilon$ admits a left inverse (i.e., a $\tilde{T} \in L(Y, X)$ with $\tilde{T}(T_0 + T_\varepsilon) = Id_X$). Then multiplication is locally open at (S_0, T_0).*

(ii) *If S_0 is such that for every $\varepsilon > 0$ there exists $S_\varepsilon \in L(Y, Z)$ with $||S_\varepsilon|| \le \varepsilon$ such that $S_\varepsilon T_0 = 0$ and $S_0 + S_\varepsilon$ admits a right inverse (i.e., an $\tilde{S} \in L(Z, Y)$ with $(S_0 + S_\varepsilon)\tilde{S} = Id_Z$), then multiplication is locally open at (S_0, T_0).*

(iii) *If T_0 admits a left inverse or S_0 admits a right inverse, then multiplication is locally open at (S_0, T_0).*

(iv) *Suppose that for every $\varepsilon > 0$ there exists T_ε with $||T_\varepsilon|| \le \varepsilon$ (resp. S_ε with $||S_\varepsilon|| \le \varepsilon$) such that $T_0 + T_\varepsilon$ has a left inverse (resp. $S_0 + S_\varepsilon$ has a right inverse). Then multiplication is locally open at $(0, T_0)$ (resp. at $(S_0, 0)$).*

Proof. (i) For a given $\varepsilon > 0$ put $T := T_\varepsilon$ and $\delta := \varepsilon/(||\tilde{T}|| + 1)$, where $\tilde{T}$ is a left inverse of $T_0 + T_\varepsilon$. For $R \in L(X, Z)$ with $||R|| \leq \delta$ define $S := R\tilde{T}$. Then $||S|| \leq \varepsilon$ and

$$(S_0 + S)(T_0 + T) = S_0 T_0 + R\tilde{T}(T_0 + T_\varepsilon) = S_0 T_0 + R.$$

The proof of (ii) is similar, and (iii) and (iv) follow immediately from (i) and (ii). □

Now we will restrict ourselves to the case of finite dimensional spaces. We will consider normed spaces X, Y and Z that are n-, k- and m-dimensional, respectively, and $T_0 : X \to Y$ as well as $S_0 : Y \to Z$ are fixed linear maps. When is $(S, T) \to ST$ locally open at (S_0, T_0) ? Our strategy will be as follows:

> We will find conditions that will allow us to apply Lemma 2.1(i) and (ii) for the proof of local openness. And if both conditions are violated we will prove that multiplication is *not* locally open.

We begin our investigations with some preparations. Denote by s and t the ranks of S_0 and T_0, respectively; we put $t_2 = \dim(\operatorname{range} T_0 \cap \ker S_0)$, and $t_1 := t - t_2$.

Lemma 2.2. *Let U be a subspace of Y.*

(i) *There exists $T \in L(X, Y)$ with range $T \subset U$ such that $T_0 + \alpha T$ is one-to-one for every $\alpha \neq 0$ iff $n - t + t_U \leq \dim U$; here $t_U := \dim(U \cap \text{range}\, T_0)$.*
(ii) *There is an $S \in L(Y, Z)$ with $S|_U = 0$ such that $S_0 + \alpha S$ is onto for every $\alpha \neq 0$ iff $s_U := \dim(U \cap \ker S_0) \leq k - m$.*

Proof. (i) Suppose that such a T exists. $\ker T_0$ is $(n - t)$-dimensional so that we may choose a basis $x_1, \ldots, x_{n-t}$ of $\ker T_0$. Let $y_{n-t+1}, \ldots, y_{n-t+t_U}$ be a basis of $U \cap \operatorname{range} T_0$, and $x_{n-t+1}, \ldots, x_{n-t+t_U}$ are vectors such that $T_0(x_i) = y_i$ for $i = n-t+1, \ldots, n-t+t_U$. It is easy to see that $x_1, \ldots, x_{n-t+t_U}$ are linearly independent so that we may select further $t - t_U$ vectors $x_{n-t+t_U+1}, \ldots, x_n$ to arrive at a basis of X.

T_0+T is one-to-one so that the $(T_0+T)(x_1), \ldots, (T_0+T)(x_n)$ are linearly independent. The range of T lies in U, and U contains the $n - t + t_U$ linearly independent vectors $T(x_1), \ldots, T(x_{n-t}), y_{n-t+1} + T(x_{n-t+1}), \ldots, y_{n-t+t_U} + T(x_{n-t+t_U})$. Thus $n - t + t_U \leq \dim U$.

Now suppose that $n - t + t_U \leq \dim U$. This inequality implies that we may choose an $(n - t)$-dimensional subspace W of U such that $W \cap \operatorname{range} T_0 = \{0\}$. Let $I : \ker\ T_0 \to W$ be a linear bijection and $V \subset X$ a complementary subspace of $\ker T_0$.

We define $T \in L(X, Y)$ by $D(x + v) := I(x)$ for arbitrary $x \in \ker T_0$ and $v \in V$. We claim that $T_0 + \alpha T$ is one-to-one for every scalar $\alpha \neq 0$. In fact, if $0 = (T_0 + \alpha T)(x + v) = T_0(v) + \alpha I(x)$ (where $x \in \ker\ T_0$ and $v \in V$), then $T_0(v) = I(x) = 0$ (since $W \cap \operatorname{range} T_0 = \{0\}$) so that $v \in V \cap \ker T_0 = \{0\}$ and $x = 0$. Thus $x + v = 0$.

(ii) Whereas an approach similar to the ideas in the proof of (i) is possible we prefer to use a duality technique that will also be important later. As usual we denote by X' the dual of a Banach space X, and for $T \in L(X, Y)$ the dual map $T' \in L(Y', X')$ is defined by $f \mapsto f \circ T$. For a subspace $U \subset X$ the annihilator $\{f \in X' \mid f|_U = 0\}$ is denoted by $U^\perp$.

The existence of an $S \in L(X, Y)$ with $S|_U = 0$ such that $S_0 + \alpha S$ is onto for $\alpha \neq 0$ is equivalent with the existence of an $S' \in L(Y', X')$ with $\operatorname{range} S' \subset U^\perp$ such that $S_0' + \alpha S'$ is one-to-one for these α. Part (i) of the lemma provides a criterion, we only have to check the new meaning of the numbers that are used there.

- The number t is the rank of S_0', it has to be replaced by s.
- The dimension of $U^\perp$ is $k - \dim U$.
- We have $(\operatorname{range} S_0')^\perp = \ker S_0$, and $(\ker S_0)^\perp \cap U^\perp = (\ker S_0 + U)^\perp$. And since $\dim(V_1 + V_2) = \dim V_1 + \dim V_2 - \dim(V_1 \cap V_2)$ for subspaces V_1, V_2 we conclude that

$$\dim(U^\perp \cap \operatorname{range} S_0') = k - [(k - s) + \dim U - s_U] = s - \dim U + s_U.$$

- The number n in (i) has to be replaced by m.

In this way our characterization of (i) translates to $m - s + (s - \dim U + s_U) \leq k - \dim U$, or $s_U \leq k - m$. $\square$

Corollary 2.3.

(i) *There exists $T \in L(X, Y)$ with $S_0 T = 0$ such that $T_0 + \alpha T$ is one-to-one for every $\alpha \neq 0$ iff $n - t_1 \leq k - s$.*
(ii) *There exists $S \in L(Y, Z)$ such that $S T_0 = 0$ and $S_0 + \alpha S$ is onto for every $\alpha \neq 0$ iff $t_2 \leq k - m$.*

Proof. One only has to apply the preceding lemma with $U = \ker S_0$ (for the proof of (i)) and $U = \operatorname{range} T_0$ (for the proof of (ii)). $\square$

So far we have avoided to use matrix representations of the maps under consideration. However, in the proof of our main theorem it will be necessary to switch between the original space and the dual space, and our argument will be much clearer if we only have to pass from columns to rows of a matrix.

In the next lemma we describe the “best possible” matrix representation of S_0 and T_0 for our purposes. One should note that the transition to a matrix representation of a given linear map might change the operator norm of this map. But this will cause no difficulties here since the problem we are interested in only depends on the topologies of the operator spaces under consideration.

Lemma 2.4. *We adopt the preceding notation and we suppose that $s \geq t$. Then there are bases in X, Y, Z, respectively, such that S_0 and T_0 have matrix representations of the following form:*

$$S_0 = \left(\begin{array}{c|c} E_s & Z_{s,k-s} \\ \hline Z_{m-s,s} & Z_{m-s,k-s} \end{array}\right), \quad T_0 = \left(\begin{array}{c|c|c} E_{t_1} & Z_{t_1,t_2} & Z_{t_1,n-t} \\ \hline Z_{s-t_1,t_1} & Z_{s-t_1,t_2} & Z_{s-t_1,n-t} \\ \hline Z_{t_2,t_1} & E_{t_2} & Z_{t_2,n-t} \\ \hline Z_{k-t_2-s,t_1} & Z_{k-t_2-s,t_2} & Z_{k-t_2-s,n-t} \end{array}\right).$$

Here $Z_{a,b}$ denotes the $a \times b$ zero matrix and E_a stands for the a-dimensional identity matrix. Obviously, S_0T_0 then has the form

$$S_0T_0 = \left(\begin{array}{c|c} E_{t_1} & Z_{t_1,n-t_1} \\ \hline Z_{s-t_1,t_1} & Z_{s-t_1,n-t_1} \\ \hline Z_{m-s,t_1} & Z_{m-s,n-t_1} \end{array}\right).$$

Proof. Write Y as the direct sum of four subspaces Y_1, Y_2, Y_3, Y_4, where

- Y_1 is the t_2-dimensional subspace range $T_0 \cap \ker S_0$;
- $Y_2 \subset \ker S_0$ is such that ker S_0 is the direct sum of Y_1 and Y_2 (this subspace is $(k - s - t_2)$-dimensional);
- Y_3 is chosen such that range T_0 is the direct sum of Y_1 and Y_3 (this is a t_1-dimensional subspace);
- finally, Y_4 is a complement of $Y_1 + Y_2 + Y_3$ in Y (an $(s - t_1)$-dimensional space). Here we make use of the fact that $s \geq t$.

Next we choose:

- $y_1, \ldots, y_{t_1}$, a basis of Y_3;
- $y_{t_1+1}, \ldots, y_s$, a basis of Y_4;
- $y_{s+1}, \ldots, y_{s+t_2}$, a basis of Y_1;
- $y_{s+t_2+1}, \ldots, y_k$, a basis of Y_2.

The first $x_1, \ldots, x_t$ are determined such that $T_0x_1 = y_1, \ldots, T_0x_{t_1} = y_{t_1}$ and $T_0x_{t_1+1} = y_{s+1}, \ldots, T_0x_{t_1+t_2} = y_{s+t_2}$. Then the $x_1, \ldots, x_t$ are linearly independent, and we choose vectors $x_{t+1}, \ldots, x_n \in \ker T_0$ such that $x_1, \ldots, x_n$ are a basis of X.

It remains to choose a basis $z_1, \ldots, z_m$ in Z. The first $z_1, \ldots, z_s$ are defined by $z_i := S_0y_i$, and the remaining ones are arbitrary. (Note that S_0 is one-to-one on $Y_3 + Y_4$.)

Then the matrix representations are as desired. □

Up to now we have provided no example where multiplication is *not* locally open. For trivial reasons this happens at all pairs (S_0, T_0) if $m, n > k$: surely one can find $R \in L(X, Z)$ with arbitrarily small $||R||$ such that $S_0T_0 + R$ has rank bigger than k, but the rank of a map $(S_0 + S)(T_0 + T)$ is always bounded by k. We will now present a less trivial example where local openness also fails. A generalization of the proof of this fact will be the essential ingredient in the verification of our main result in Theorem 2.5.

Example. We will consider the case $n = k = m = 4$, and S_0, T_0 are defined by the following matrices:

$$S_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad T_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad S_0T_0 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Here $s = t = 2$ and $t_1 = t_2 = 1$. We claim that multiplication is not locally open at (S_0, T_0). To this end we have to provide a positive ε_0 such that for arbitrarily small δ it is not possible to write all $S_0T_0 + R$ with $||R|| \leq \delta$ in the form $(S_0 + S)(T_0 + T)$, where $||S||, ||T|| \leq \varepsilon_0$.

We choose $\varepsilon_0 > 0$ such that the following two conditions are satisfied:

- For each T with $||T|| \leq \varepsilon_0$ the second column of $T_0 + T$ is not the zero column.
- For each S with $||S|| \leq \varepsilon_0$ the first two rows of $S_0 + S$ are linearly independent.

Surely such ε_0 exist. (In order to avoid to work with the operator norm of matrices we are not going to define ε_0 explicitly.) Now let $\delta > 0$ be arbitrary. We claim that the definition

$$R = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix}, \quad \text{i.e.,} \quad S_0T_0 + R = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \delta & 0 \\ 0 & 0 & 0 & \delta \end{pmatrix},$$

leads to an R with the desired properties.

Suppose that there were S, T with $||S||, ||T|| \leq \varepsilon_0$ and $(S_0 + S)(T_0 + T) = S_0T_0 + R$: we will show that this leads to a contradiction.

The second column of $S_0T_0 + R$ is zero, but the second column of $T_0 + T$ is nontrivial by our choice of ε_0. Hence $S_0 + S$ has a nontrivial kernel and therefore it cannot have full rank. We may choose scalars a_1, a_2, a_3, a_4 with $(a_1, a_2, a_3, a_4) \neq (0, 0, 0, 0)$ such that $(a_1, a_2, a_3, a_4)(S_0 + S) = (0, 0, 0, 0)$. (I.e., $\sum_i a_i r_i = 0$, where r_1, r_2, r_3, r_4 denote the rows of $S_0 + S$.)

It is not possible that $a_3 = a_4 = 0$ since r_1, r_2 are linearly independent due to the choice of ε_0 so that $(a_3, a_4) \neq (0, 0)$.

From $(a_1, a_2, a_3, a_4)(S_0 + S) = (0, 0, 0, 0)$ it follows that

$$(0, 0, 0, 0) = (a_1, a_2, a_3, a_4)(S_0 + S)(T_0 + T) = (a_1, a_2, a_3, a_4)(S_0T_0 + R).$$

Or, to state it differently, $\sum_i a_i r'_i = (0, 0, 0, 0)$, with $r'_i =$ the i'th row of $S_0T_0 + R$. But this is surely not the case since the last two entries of $\sum_i a_i r'_i$ are $a_3\delta, a_4\delta$, and $(a_3, a_4) \neq (0, 0)$.

Here is our main result:

Theorem 2.5. *Let X, Y, Z be real or complex normed spaces with dimensions n, k and m, respectively. $S_0 \in L(Y, Z)$ and $T_0 \in L(X, Y)$ are linear operators, we denote by s resp. t the rank of S_0 resp. the rank of T_0. The number t is written as $t = t_1 + t_2$, where t_2 is the dimension of $\mathrm{range}\, T_0 \cap \ker S_0$.*

The following conditions are equivalent:

(i) *Multiplication is locally open at (S_0, T_0).*
(ii) *$t_2 \leq k - m$, or $n - t_1 \leq k - s$.*
(iii) *There exists $T \in L(X, Y)$ with $S_0T = 0$ such that $T_0 + \alpha T$ is one-to-one for every $\alpha \neq 0$, or there exists $S \in L(Y, Z)$ such that $ST_0 = 0$ and $S_0 + \alpha S$ is onto for every $\alpha \neq 0$.*

Proof. The equivalence of (ii) and (iii) was shown in Corollary 2.3, and (iii) $\Rightarrow$ (i) is a consequence of Lemma 2.1(i) and (ii). One only has to note that a linear map between finite dimensional spaces has a left inverse (resp. a right inverse) iff it is one-to-one (resp. onto). So it remains to prove that (i) implies (ii).

Assume that $t_2 > k - m$ and $n - t_1 > k - s$: we will show that multiplication is not locally open at (S_0, T_0). We will adopt the strategy that worked successful in the preceding example.

Let us first assume that $s \geq t$. We will work with the matrix representations of S_0 and T_0 from Lemma 2.4. Our ε_0 is chosen such that:

- If S is an $m \times k$-matrix such that $||S|| \leq \varepsilon_0$, then the first s rows of $S_0 + S$ are linearly independent.
- If T is a $k \times n$-matrix with $||T|| \leq \varepsilon_0$, then the columns of $T_0 + T$ at positions $t_1 + 1, \ldots, t$ are linearly independent.

Let $\delta > 0$ be arbitrary. We will define R such that $||R|| \leq \delta$, but $S_0T_0 + R$ cannot be written as $(S_0 + S)(T_0 + T)$ with $||S||, ||T|| \leq \varepsilon_0$. The idea will be to have nonzero entries of R only in the bottom right block of S_0T_0.

Let us first analyze how many rows are there below the E_s-block in S_0. The kernel of S_0 is $(k - s)$-dimensional, and there is at least a t_2-dimensional subspace in this kernel. Thus $k - s \geq t_2$, or $s \leq k - t_2$. But $k - t_2 < m$, and consequently we find at least $p := k - t_2 + 1 - s$ rows below E_s. The inequality $n - t_1 > k - s$ yields $n - t \geq p$, and this enables us to find a $p \times (n - t)$-matrix $\tilde{R}$ with linearly independent rows.

We now define R: all entries are zero, unless the entries in the rows at positions $s+1, \ldots, s+p$ between the columns $t+1, \ldots, n$ where we insert the entries of $\hat{R} := \tau \tilde{R}$; here τ is chosen so small that $||R|| \leq \delta$.

We now proceed as in the preceding example. Suppose that one could write $S_0T_0 + R$ as $(S_0 + S)(T_0 + T)$ with $||S||, ||T|| \leq \varepsilon_0$. The linearly independent columns of $T_0 + T$ at positions $t_1 + 1, \ldots, t$ are mapped to zero by $S_0 + S$ since the corresponding columns of $S_0T_0 + R$ are zero. Hence the kernel of $S_0 + S$ is at least t_2-dimensional so that the rank of this matrix is at most $k - t_2$. We conclude as above that each $k - t_2 + 1$ rows of $S_0 + S$ must be linearly dependent so that we may choose a nontrivial linear combination $\sum_{i=1}^{k-t_2+1} a_i r_i = 0$. (By $r_1, \ldots, r_m$ we denote the rows of $S_0 + S$.) But $r_1, \ldots, r_s$ are linearly independent by our choice of ε_0 so that there must be nonzero numbers among the $a_{s+1}, \ldots, a_{s+p}$. With $r'_i =$ the rows of $S_0T_0 + R$ we conclude (with an argument as in the special four-dimensional case that was discussed just before the theorem) that $\sum_{i=1}^{k-t_2+1} a_i r'_i = 0$. But this is not possible since it would imply that a nontrivial linear combination of the rows of $\hat{R}$ is zero.

It remains to deal with the case $s < t$. As in the proof of Lemma 2.2(ii) we will reduce it to the case $s > t$ by using duality. Suppose that we assume (i). If multiplication is locally open at (S_0, T_0), then it is locally open also at (T'_0, S'_0) since $T \mapsto T'$ is an isometrical isomorphism that reverses the order of multiplication. But the rank of T'_0 is larger than the rank of S'_0 so that – by the first part of the proof – we know that one of the inequalities

$$t'_2 \leq k' - m', \quad n' - t'_1 \leq k' - s'$$

holds. Here the numbers with the prime stand for the corresponding parameters for the new situation:

- $k' = k$, $m' = n$, $n' = m$, $t' = s$, $s' = t$.
- $t'_2 = \dim(\text{range}\, S'_0 \cap \ker\ T'_0) = s - t_1$ (cf. the proof of Lemma 2.2(ii)).
- $t'_1 = t' - t'_2 = s - (s - t_1) = t_1$.

Thus $t'_2 \leq k' - m'$ in fact means $s - t_1 \leq k - n$, and $n' - t'_1 \leq k' - s'$ is the same as $m - t_1 \leq k - t$, or $t_2 \leq k - m$. $\square$

3. Multiplication in the Banach algebra $\mathcal{M}_n$

We now consider the special case $k = n = m$:

Theorem 3.1. *Let $S_0, T_0 \in \mathcal{M}_n$ be given. s and t stand for the rank of S_0 and T_0, respectively. t_2 is the dimension of range $T_0 \cap$ ker S_0, and $t_1 := t - t_2$.*

(i) *Suppose that $s \geq t$. Then multiplication in $\mathcal{M}_n$ is locally open at (S_0, T_0) iff $t_2 = 0$ iff there exist $S \in \mathcal{M}_n$ with $ST_0 = 0$ such that $S_0 + \alpha S$ is invertible for every $\alpha \neq 0$.*

(ii) *In the case $s \leq t$ multiplication in $\mathcal{M}_n$ is locally open at (S_0, T_0) iff $s \leq t_1$ iff there exist $T \in \mathcal{M}_n$ with $S_0 T = 0$ such that $T_0 + \alpha T$ is invertible for every $\alpha \neq 0$.*

Proof. (i) If $s \geq t$ and $k = n = m$ the condition $s \leq t_1$ implies $t_2 = 0$. Thus the assertion follows from Theorem 2.5 and Corollary 2.3.

(ii) Here one only has to note that in the case $s \leq t$ the condition $t_2 = 0$ yields $s \leq t_1$ so that Theorem 2.5 and Corollary 2.3 yield the result. □

Remark. If (under the assumption $k = n = m$) $s = t$ holds the conditions $t_2 = 0$ and $s \leq t_1$ are equivalent, and if they are satisfied one has both invertible $S_0 + \alpha S$ and invertible $T_0 + \alpha T$.

We supplement our theorem with some further results concerning local openness of multiplication in $\mathcal{M}_n$:

Proposition 3.2. *Let $S_0, T_0 \in \mathcal{M}_n$ be given.*

(i) *If multiplication is locally open at (S_0, T_0) it does not follow that it is locally open at (T_0, S_0).*
(ii) *If multiplication is locally open at (S_0, T_0) then it is also locally open at $(T_0^\top, S_0^\top)$; here $S^\top$ stands for the transpose of a matrix S.*
(iii) *Suppose that $\mathbb{K}^n$ is provided with the usual scalar product and that S^* denotes the adjoint of a matrix S. Then multiplication is locally open at (S_0, T_0) iff it is locally open at (T_0^*, S_0^*). In particular multiplication is locally open at (S_0, T_0) if it is locally open at (T_0, S_0) for self-adjoint S_0, T_0.*
(iv) *The pairs where multiplication is locally open are dense in $\mathcal{M}_n \times \mathcal{M}_n$.*

Proof. (i) As a simple example where this can happen consider

$$S_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad T_0 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}.$$

We have $\ker S_0 = \operatorname{range} S_0 = \ker T_0 = \{0\} \times \mathbb{K}$ and $\operatorname{range} T_0 = \mathbb{K} \times \{0\}$. Thus $s = t = 1$, and Theorem 3.1 implies the assertion.

(ii) and (iii) follow from the observation that $S \mapsto S^\top$ (resp. $S \mapsto S^*$) are isometrical isomorphisms that reverse the order of multiplication.

(iv) is a consequence of the fact than invertible matrices are dense in $\mathcal{M}_n$. □

Up to now we have no satisfactory characterization of local openness for the case of more general Banach algebras or multiplication in operator spaces. (Cf. the last section of [4] for the discussion of an example.) The formulation and the proof of our main theorem was strongly dependent on the fact that we are dealing with *finite-dimensional* spaces: finiteness of the rank; a one-to-one operator has a right inverse; $\dim(\text{range}\, S) + \dim(\ker S) = k$ for operators on a k-dimensional space; the space has the same dimension as its dual etc.

Parts of our results have a counterpart for continuous linear operators on a separable Hilbert space, but we are still far from an understanding how an interplay of properties of the ranges, the kernels and the spectra gives rise to local openness.

References

[1] M. Balcerzak, W. Wachowicz, W. Wilczyński, Multiplying balls in the space of continuous functions on $[0, 1]$, Studia Math. 170 (2) (2005) 203–209.
[2] M. Balcerzak, E. Behrends, F. Strobin, On certain uniformly open multilinear mappings, Banach J. Math. Anal. 10 (2016) 482–494.
[3] E. Behrends, Walk the dog, or: products of open balls in the space of continuous functions, Funct. Approx. 44 (1) (2011) 153–164.
[4] E. Behrends, Products of n open subsets in the space of continuous functions on $[0, 1]$, Studia Math. 204 (1) (2011) 73–95.
[5] E. Behrends, Where is pointwise multiplication on CK-spaces locally open?, Fund. Math. 236 (2017) 51–69.
[6] A. Komisarski, A connection between multiplication in CX and the dimension of X, Fund. Math. 189 (2) (2006) 149–154.
[7] A. Wachowicz, Multiplying balls in $C^{(N)}[0, 1]$, Real Anal. Exchange 34 (2) (2008) 445–450.